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Heterotic Flux Compactifications with Sasakian Manifolds

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Abstract

The main concerns of this thesis are the construction of 6-dimensional manifolds with $SU(3)$-structure as well as finding instantons on these spaces.
We begin by giving a brief exposition of how both play a role in heterotic flux compactifications. Here the internal geometry is required to carry an $SU(3)$-structure defined by a Killing spinor, and the gauge field background has to be an instanton with respect to this spinor if the compactifications is to preserve supersymmetry.
Chapters 2, 3 and 4 are devoted to the relevant mathematical framework. First, we elaborate on the geometry and classification of G-structures, their relation to nowhere-vanishing, or invariant, sections of associated vector bundles, and connections on the respective principal bundles in chapter 2.
Chapter 3 then introduces several types of $SU(2)$- and $SU(3)$-structures in five and six dimensions, respectively. We include a section on Sasakian geometry and consider a family of different types of $SU(2)$-structures on S^5 as an example.
The mathematical background is completed by a treatise on the instanton condition and different ways of implementing it in chapter 4. We investigate the behavior of a G-structure upon right-action with basepoint-dependent elements of $Gl(D,\mathbb{R})$ and classify which of these transformations again yield a G-structure. Subsequently, we provide an answer as to when a pair of G-structures related by such a deformation induces the same instanton conditions.
In chapter 5 we introduce a general method to obtain $SU(3)$ 6-manifolds from 5-dimensional manifolds with families of $SU(2)$-structures. We then show how the deformations considered in chapter 4 can be combined with this construction. Starting from Sasaki-Einstein 5-manifolds M^5, we thereby obtain Kähler-torsion and nearly Kähler $SU(3)$-structures on sine-cones over M^5 as well as half-flat $SU(3)$-structures on the cylinder over M^5. In an appropriate infinite-volume limit, both sine-cones approach the Calabi-Yau metric cone over the Sasaki-Einstein space.
Chapter 6 starts with a formalization of the procedure introduced in [1], which allows us to reduce the instanton equations to matrix equations for a perturbation of a given instanton. Using suitable ansätze, we first reproduce solutions of [2] for the Kähler-torsion sine-cones and argue that these will be present in all our constructions. On the nearly Kähler sine-cones, we at first obtain these solutions for the perturbation only, but we then find another $\mathfrak{su}(2)$-valued instanton on that space. The computation of the nearly Kähler canonical connection then provides us with a suitable ansatz for extensions of that second instanton. Thereby, we independently prove that the canonical nearly Kähler connection is an instanton and obtain another solution. On the half-flat cylinders, we obtain solutions with constant perturbation only, which correspond to new instantons living on M^5 rather than on the cylinder.

Contents

Chapter 1

String Theory and Geometry

1.1 Introduction

Interest in string theory has led to vast progress in several areas of current research, among which itself is only one of many. String theory has repeatedly pointed out where the frontiers of our present insight into physics and mathematics are located and often even how these may be extended. However, from a physicists point of view, string theory at its present stage is a purely hypothetical concept. There is no feature of this framework which could be used in order to experimentally differentiate between string theory and the somewhat disjoint union of quantum field theory of elementary particle physics and general relativity, the two pillars of todays research in high-energy physics. This is mostly because the discrepancies between these frameworks would become apparent only at very high energies not accessible in experiments at this point.

While much is known about the roof that is supported by these pillars, there is no conclusive answer as to what mutual foundation they rest on. Nevertheless, string theory appears to be a good candidate for such a unifying concept. It is an extraordinarily rich framework, allowing for intuitive limits in which it reproduces the physics of elementary particles and general relativity, while avoiding their shortcomings in high-energy regimes. Also, on the purely mathematical side, it has shown to be fruitful, as the theory is so intricate that at several points physical research implies that there exist ways in certain directions, which one cannot really uncover before improving the mathematical toolkit. New techniques drawn from research in string theory keep having important applications to statistical and particle physics, differential and algebraic geometry and topology, and many more active fields of current research.

In order to understand the physics behind a theoretical model, it has repeatedly proven valuable to find explicit solutions to its equations. These can then be analyzed and interpreted, and have often led to deeper insights into the physical implications and scope of the theory. Some of the most prominent examples are the Lienert-Wiechert potentials in electromagnetism, Onsager's solution to the Ising model, the Schwarzschild and FLRW metrics in general relativity, and perhaps most recently the $AdS_5 \times S^5$ solutions of type II string theory.

Here we will work towards explicit solutions to heterotic supergravity, the low-energy effective theory of the heterotic string. In particular, we will focus on the instanton equation, which is one of the conditions for unbroken supersymmetry, and whose so-

lutions have been successfully extended to solutions of the complete set of heterotic supergravity equations for example in [2–4].
Although motivated by the physical framework of string theory, the problem we are considering is of mathematical and, in particular, geometrical nature. Therefore, we choose to employ a very mathematical style of writing and aim at a high level of mathematical rigor in this work.

The outline of this thesis is as follows: In chapter 1 we will first review the subject of string compactifications very briefly, thereby introducing the relevant equations. We see that the presence of a generator of supersymmetry imposes certain restrictions on the geometry of the spacetime background M. More precisely, it implies the existence of a G-structure on M.
Therefore, we explore G-structures and some of their geometric features in chapter 2. We emphasize the point of view of principal fiber bundles, which are the very foundation of the mathematics we need to pursue our goal of finding solutions to the instanton equations. After introducing several different characterizations of G-structures, we turn to connections on the principal bundles that the G-structures are subbundles of and review the classification idea of Gray and Hervella [5] in terms of such connections.
We review the definition of $SU(2)$-structures in 5 and $SU(3)$-structures in 6 dimensions, as well as some of their geometric features, in chapter 3. Additionally, we provide a brief overview of Sasakian geometry.
Chapter 4 concludes the part on the mathematical background by elaborating how G-structures induce instanton conditions and how these can be formulated in different ways. In the last section of that chapter, we consider certain deformations of G-structures and prove a classification (proposition 4.3.1) of the subclass of these deformations that leads from a given G-structure to a new one. In an extra step, we derive an algebraic constraint on the deformations which is equivalent to the property that the original and the new G-structure define precisely the same instantons in corollary 4.3.5. In particular, G-structures related by deformations of the considered type have precisely the same instanton moduli spaces.
We begin chapter 5 by giving a general prescription of how one can obtain several $SU(3)$-structure 6-manifolds from $SU(2)$-structure 5-manifolds in proposition 5.1.1. The combination of this result with the deformations of G-structures investigated in chapter 4 enables us to explicitly construct several types of G-structures on conical extensions of a given manifold. We apply this procedure to Sasaki-Einstein 5-manifolds and obtain Kähler-torsion and nearly Kähler $SU(3)$-structures on sine-cones as well as half-flat $SU(3)$-structures on cylinders over these spaces.
Chapter 6, as the final chapter of this thesis, is devoted to the construction of instantons on these spaces. First, culminating in proposition 6.1.4, we formalize a procedure introduced in [1]. This procedure reduces the instanton condition on

gauge fields to a set of matrix equations on a perturbation of a given instanton. Subsequently, we carry out this reduction in several situations on the spaces constructed in chapter 5. Here the formal results obtained throughout this thesis find numerous applications and endow us with immediate insight into subtle details of the mathematical problems under consideration.

1.2 The Five String Theories

We begin with a very compact tour from some of the basics of string theory to the point where this thesis is embedded into current research. Therefore, this chapter is a merging of different introductory texts on string theory, mostly taken from the textbooks [6] and [7].

In contrast to quantum field theory, in string theory the fundamental building blocks of matter are assumed to be strings. These are one-dimensional objects of very small length. Strings are assumed to move through spacetime, their motion being governed by the Polyakov action [6]

$$S_P[h, X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h_{..}}\; h^{\alpha\beta} \left(\partial_\alpha X^\mu\right) \left(\partial_\beta X^\nu\right) g_{\mu\nu}(X). \tag{1.2.1}$$

Here $X : \Sigma \to M^D$ is an embedding of the string world sheet Σ into the spacetime manifold M^D, g is a Lorentzian metric on M, and h is a Lorentzian metric on Σ, an auxiliary field. The notation $h_{..}$ indicates that the metric on TM^D is to be used, rather than the metric it induces on T^*M^D. $\alpha' = \frac{1}{2\pi T}$ is called the Regge slope, where T is the tension of a string. Due to the X-dependence of g, this action is highly non-polynomial. However, assuming M^D to be Minkowski space one can transform this action to a polynomial one by choosing the conformal gauge.
Viewing the X^μ as massless fields on the world sheet, they are seen to satisfy the wave equation., whence the decompose into left- and right-moving parts. By canonical quantization of their Fourier modes, one obtains creation and annihilation operators. This points out a very important duality in string theory. The quantities on the world sheet can on the one hand be viewed as observables on spacetime, as for example coordinates of M^D. On the other hand, however, they may be considered as quantum fields on Σ. One can then interpret these geometrically as measuring certain quantities assigned to points of the string world sheet, as for example the position of a certain point $\sigma \in \Sigma$ in spacetime.
In analogy to the picture of quantum field theory, strings are assumed to be able to appear at arbitrary points in spacetime, and their excitations can then be interpreted as excitations of effective quantum fields. (This is a picture which would actually require a string *field* theory, which is an active field of current research. For several results and further references see e. g. [8, 9].)

The analysis of the particle spectrum of the above theory leads to remarkable and, yet, peculiar conclusions. We list just the most important ones:

- The theory necessarily contains a tachyonic particle, i. e. one with negative mass square. This cannot be interpreted in a phenomenologically consistent way.
- Since the quantum fields on Σ are purely bosonic, the spacetime theory does not contain fermionic particles.
- Different ways of quantizing the action (1.2.1) coincide for $D = 26$ only, whereas we have observed four spacetime dimensions so far.
- Most remarkably, the spectrum contains a massless particle of helicity 2, whose interactions in the emergent spacetime field theory coincide with those induced by the Einstein-Hilbert action up to corrections of order α'.

The last property implies that general relativity is emergent in the effective spacetime field theory of the quantized string theory up to corrections suppressed by α'. While this may seem surprising, it may be due to the fact that in two dimensions, as on the string world sheet, the Einstein-Hilbert action is topological. Hence it shifts the action by a constant only. This means that the Polyakov action (1.2.1) is in fact the full theory of D real massless scalar fields living on Σ, coupled to a gravitational field. Being trivial, the Einstein-Hilbert part of the action can of course be quantized in two dimensions. Here one takes for granted that quantization is consistent with the assumption that it is always possible to go to the conformal gauge. That this holds true is more apparent in path integral quantization, where all fields are treated as classical ones in calculations.

Considering the quantum field theory on the world sheet as fundamental, one may alter this two-dimensional model for example by the inclusion of supersymmetry. One can introduce either global supersymmetry on the world sheet fields (RNS formalism) or local supersymmetry on the target space coordinates (GS formalism) [10, 11]. Either way this adds new fermionic fields to the world sheet, which in turn add fermionic excitations to the spectrum of the effective field theory of the bosonic string.

Thereby, the fermion problem of the bosonic string gets cured. However, the tachyon problem must be addressed by hand. Either one has to eliminate half the fermionic degrees of freedom independently in the left- and right-moving sectors of the RNS string, or one has to choose a chirality of the anticommuting coordinates on the target space. Both of these procedures remove the tachyonic state from the spectrum.

The dimension problem is affected, but not solved by the inclusion of supersymmetry. The critical dimension of superstring theories turns out to be $D = 10$.

Of these superstring theories there exist three versions, namely:

- Type IIB: Oriented open and closed superstrings with the same chiralities in the right and left-moving sectors,

- Type IIA: Oriented open and closed superstrings with opposite chiralities in the right and left-moving sectors,
- Type I: Unoriented open and closed superstrings.

These are not the only consistent versions of superstring theory. There have been constructed two so-called heterotic superstring theories, which contain closed strings only. In contrast to the aforementioned theories, their left-moving sector consists of the left-moving sector of the $D = 26$ bosonic string with 16 of the bosonic coordinates compactified on a torus. Their right-moving sector coincides with that of the $D = 10$ superstring. The name heterotic stems from this asymmetry of the right and left-moving sectors. The special feature of these theories is a local world sheet symmetry in the 16 compactified bosonic coordinates. Generically, this is a global $U(1)^{16}$-symmetry, but for special radii of the internal torus it gets enlarged to non-Abelian symmetry groups. There are only two such symmetry enhancements possible, and the resulting theories are named after the Lie groups of these global symmetries. They are

- H($E_8 \times E_8$): Heterotic superstrings with symmetry group $E_8 \times E_8$,
- H($SO(32)$): Heterotic superstrings with symmetry group $SO(32)$.

All of these five known versions of superstring theories give rise to effective field theories in their perturbative expansions. These differ in certain features, as for example in their particle spectra. Despite this, the five string theories turn out to be connected by certain non-perturbative dualities, leading to the conjecture that there exists an undiscovered framework of which the string theories are certain perturbative sectors.
As we are still to explore even the low-energy phases of this framework, the concern of this thesis lies within a particular branch of the string theories, namely the heterotic theories. For this reason, we focus on heterotic strings in the remainder of this chapter.

1.3 Low-Energy Effective Theory of the Heterotic String

As explained in the previous section, strings admit certain modes of oscillation, which appear as particles in the effective theory of the string theory under consideration. That is, one assumes strings to potentially provide excitations at any point of spacetime, just as a field would do. In particular, on a length scale which is macroscopic compared to the string length strings will effectively behave like point particles. In other words, the high-energy phase of string theory changes to a phase of effective point particles in low-energy regimes, and an effective description in terms of qutnum fields becomes appropriate.

In these regimes only the massless excitations of strings have to be taken into account for the particle content of the effective field theories. Nevertheless, one has to keep in mind that all excitations are relevant in the string theory computations of scattering processes of string excitations. Here, massive modes still occur as internal legs in scattering processes. Their contributions to scattering processes affect the effective field theory, thus giving rise to stringy corrections of low-energy physics. In order to arrive at a proper field theory, one then needs an effective Lagrangian for the massless fields. This may be obtained by remodeling the interactions of the string modes that encode the effective particle content on the corresponding fields. As the effective fields arise from strings, a string of the respective theory should interact with a background of these fields. This turns out to be manifest due to the following important fact: An amplitude for the propagation of a string through a background of its effective fields can be computed in two different ways. First, one can compute it from the free string action with certain coherent insertions of vertex operators. That is, one considers a string that interacts with the background of strings which produce the respective field, and that propagates as a free string between these interactions. Second, the same amplitude can be computed from the action of a string moving through a background of just these effective fields, which are taken to interact with the quantum fields on the string world sheet. Remarkably, these two computations give precisely the same result. Moreover, as in the first picture the propagation of strings in such a background can be computed from just the standard action, i. e. the one without additional background fields, the world-sheet theory still has (super)conformal symmetry. This should then be present in the second picture as well. Again remarkably, the conditions on the background fields implied by the requirement that the conformal invariance of the world sheet theory be maintained precisely coincide with the field equations as derived from the low-energy effective spacetime actions.

The field content of the low-energy effective field theory of the heterotic string is that of $\mathcal{N} = 1$, $D = 10$ supergravity coupled to $\mathcal{N} = 1$, $D = 10$ super-Yang-Mills theory. Its bosonic sector consists of the real scalar dilaton field Φ, a gauge field A for either a $SO(32)$ or an $E_8 \times E_8$ gauge symmetry on spacetime, a metric G and a 2-form potential B_2. The fermionic sector is formed by the supersymmetry partners of the bosonic fields, which are the dilatino λ, a gaugino χ and a gravitino ψ.
As fields on spacetime, these are related via local supersymmetry transformations that can be found e. g. in [11]. We consider effects of quantum fluctuations as small, thus splitting the fields into quantum fluctuations and classical background fields. These classical fields may be considered as the vacuum expectation values of the quantum fields. Of course, all this is only consistent in the low-energy region of the theory, which is the regime we are studying.
As it is for common local symmetries, a field configuration does not break supersym-

metry spontaneously if it is invariant under the symmetry. Therefore, if we intend not to break supersymmetry at this stage, we have to require all supersymmetry variations of the fields to vanish. A great simplification of the coupled system of supersymmetry transformations is achieved by the restriction to field configurations in which all the fermionic fields vanish identically. For example, all the supersymmetry variations of the bosonic fields will vanish, as they are built from the fermionic field content.

This leaves us with the supersymmetry transformations as considered for instance in [10]:

$$\begin{aligned} \delta\,\psi &= \nabla^+\,\epsilon, \\ \delta\,\lambda &= \gamma\big(\mathrm{d}\Phi - \frac{1}{2}\,H\big)(\epsilon), \\ \delta\,\chi &= -\,\frac{1}{2}\,\gamma(F^A)(\epsilon). \end{aligned} \tag{1.3.1}$$

Here, $\gamma : \Lambda^* T^* M \to Cl(TM,g)$, $\mu^a\, e_a \mapsto \mu^a \gamma_a$ is the isomorphism from the exterior tensor algebra of the cotangent bundle of M into the Clifford bundle $Cl(TM,g)$ of (M,g), where M is the spacetime manifold. Clifford multiplication is understood as a map $Cl(TM,g) \mapsto End(\mathcal{S})$, for $\mathcal{S}$ being a spinor bundle over M. H is a field-strength 3-form subject to the anomaly cancellation condition

$$H \coloneqq \mathrm{d}B_2 + \frac{\alpha'}{4}\,\big(\mathrm{CS}(\omega^-) - \mathrm{CS}(A)\big), \tag{1.3.2}$$

where $\mathrm{CS}(\omega^-)$ and $\mathrm{CS}(A)$ denote the Chern-Simons 3-forms associated with the connections ω^- and A respectively, and F^A is the field strength of the gauge connection. The connections $\nabla^\pm$ differ from the Levi-Civita connection by a totally antisymmetric torsion term, i. e. with $\flat_g \circ T \in \Omega^3(M)$. In local coframes their connection forms are given by

$$(\omega^\pm)_{abc} = (\omega^g)_{abc} \pm \frac{1}{2}\,H_{abc}. \tag{1.3.3}$$

ϵ is the generator of supersymmetry and, therefore, has to be a nowhere-vanishing spinor field on M. As indicated above, unbroken supersymmetry requires all these variations to vanish identically.

Under the restriction of vanishing fermionic fields, only the bosonic fields are dynamical, whence the low-energy effective action contains these only. To first oder in string corrections, the bosonic sector of the low-energy effective action for the field theory emergent from the heterotic strings reads [12]

$$\begin{aligned} S = \frac{1}{2\,\kappa_{10}{}^2} \int \mathrm{d}^{10}X\,\sqrt{-g_{..}}\,e^{-2\Phi}\left(\mathrm{Sc}(g) + 4(\nabla\,\Phi)^2 - \frac{1}{2}\,|H|^2\right) \\ + \frac{1}{2\,\mathrm{G}_{10}{}^2} \int \mathrm{d}^{10}X\,\sqrt{-g_{..}}\,e^{-2\Phi}\left(\mathrm{tr}\big(|R^+|^2\big) - \mathrm{tr}\big(|F^A|^2\big)\right) + \mathcal{O}(\alpha'^2). \end{aligned} \tag{1.3.4}$$

Here g is the spacetime metric, $\mathrm{Sc}(g)$ the scalar curvature computed from it, and R^+ is the curvature of the connection ∇^+. The coupling constants are related by $\mathrm{G}_{10}{}^{-2} = \frac{\alpha'}{4}\,\kappa_{10}{}^{-2}$ [6].
The field equations that one obtains upon variation of the fields read [12]

$$
\begin{aligned}
0 &= \mathrm{Ric}(g)_{\mu\nu} + 2(\nabla^g\,\mathrm{d}\Phi)_{\mu\nu} - \frac{1}{4}\,H_{\mu\rho\lambda}\,H_\nu{}^{\rho\lambda} \\
&\quad + \frac{\alpha'}{4}\Big(R^-{}_{\mu\rho\lambda\kappa}\,R^-{}_\nu{}^{\rho\lambda\kappa} - \mathrm{tr}\big(F^A{}_{\mu\rho}\,F^A{}_\nu{}^\rho\big)\Big), \\
0 &= \mathrm{Sc}(g) + 4\,\Delta\Phi - 4\,|\mathrm{d}\Phi|^2 - \frac{1}{2}\,|H|^2 + \frac{\alpha'}{4}\,\mathrm{tr}\big(|R^-|^2 - |F^A|\big), \\
0 &= \mathrm{d}_A\big(*(e^{-2\Phi}\,F^A)\big) + *H\wedge e^{-2\Phi}\,F^A, \\
0 &= \mathrm{d}\big(*(e^{-2\Phi}\,H)\big).
\end{aligned}
\tag{1.3.5}
$$

The low-energy field theory of the heterotic string theory is often called heterotic supergravity. Having established the fundamental framework, we would like to obtain solutions to its field equations. This is desirable since they may feature special properties that do not occur in other theories of spacetime. It is possible, moreover, that important consequences of the theory may not be seen from the field equations directly, but rather become apparent from its solutions only.
In general relativity, for example, solutions like the Schwarzschild and Friedmann-Lemaître-Robertson-Walker metric have had a tremendous impact on research in this field. The former introduced the concept of event horizons, whose existence had not been considered until then. The latter, on the contrary, became the first Standard Model of cosmology, already bearing evidence for the big bang hypothesis. Thus, solutions of special type hopefully would enable us to learn more about the physical content of string theory eventually. However, general solutions will not yield phenomenologically acceptable models. We still have to refine the geometry of spacetime.

1.4 Compactification: From Calabi-Yau to Fluxes

The inclusion of supersymmetry yields a great leap forward in matching string theory with reality. Nevertheless, there remain open problems, some of which are quite conceptual in nature. It turns out that at this point one is several steps away from even phenomenologically matching observations.
First, there is the issue that supersymmetry has not been observed so far, whereas it is intrinsic to all superstring theories. This may be resolved, for example, by spontaneous symmetry breaking in the effective field theory. Furthermore, it is not inherent in the framework that the low-energy effective theory will contain the spectrum of physical particles we observe. At the electroweak scale, the effective theory

has to contain the Standard Model of elementary particles and general relativity, possibly with other particles forming dark matter.
This is connected to the third, perhaps most obvious and urgent problem, the dimensions problem. All superstring theories can be consistent on the quantum level only if the spacetime the strings live in is of dimension $D = 10$. This, of course, is in harsh contrast to the evident fact that we observe only 4 spacetime dimensions. One way out of this dilemma is to assume that in 6 of its 10 independent directions spacetime is of very small size. This procedure is called compactification. Although not the most general, a straightforward ansatz for such a spacetime is a direct product geometry $M^{10} = M^4 \times M^6$. For this to resolve the dimensions problem, the internal space M^6 has to be a compact manifold.
This also has an effect on the particle content of the effective theory. The particles we observe can be described as excitations of fields with components transforming in representations of $Spin(1,3)$. However, in all superstring theories, the string excitations are states in representations of $Spin(1,9)$. Assume a spacetime as described above was endowed with a metric g such that $TM^4 \perp_g TM^6$. This allows to choose orthonormal bases on the two factors separately Therefore the representations of $Spin(1,9)$ splits into representations of $Spin(1,3) \times Spin(6)$, thus leading us closer to the desired physical behavior.
We may then, as a first step, search for field configurations in which the only non-trivial field on M^4 is the metric. These will appear as vacuum configurations on the effective spacetime M^4. In particular, this means that any $Spin(1,3)$ field content is trivial as this comprises the field content of the effective 4-dimensional theory. Thus, the bosonic background fields Φ, F^A and H have non-vanishing components along the internal space only.
Furthermore, assume that the effective spacetime M^4 has certain symmetry properties, as for example that it is a locally symmetric space. This implies that around every $x \in M^4$ there exists an open subset $U_x \subset M^4$ together with a local isometry $s_x : U_x \to U_x$ satisfying $(s_x)_{*|x} = -\mathbb{I}_{T_xM^4}$. In order for this symmetry to be an actual feature of the effective theory in 4 dimensions, it must not be spontaneously broken by the physics in the internal, compact directions. This has the following implications: If H has non-vanishing components only in the internal directions, we obviously have $(s_x^* H)_{|x} = H_{|x}$ and also $(s_x^* \Phi)(x) = \Phi(x)$, as we take s_x to act as the identity on M^6. We could as well investigate the change of these fields in certain directions. This must then be invariant under the local isometries as well, since s_x is defined on a whole neighborhood of x. That is, we should require ${s_x}^*(\mathcal{L}_X H)_{|x} = (\mathcal{L}_X H)_{|x}$ for any $X \in T_xM^4$. However, we have ${s_x}^*(\mathcal{L}_X H)_{|x} = (\mathcal{L}_{-X} H)_{|x} = -(\mathcal{L}_X H)_{|x}$. Thus, H and by the same reasoning Φ may not depend on the macroscopic directions, if we require, for instance, locally symmetric effective spacetimes. Assuming that the gauge sector lives on a princi-

pal bundle over M^6, this must also hold true for local representations of the gauge field A and, therefore, as well for its field strength F^A and Riemannian metric g_6. As a very special case, we might search for compactifications in which the effective spacetime M^4 is Minkowski space. This is covered by the preceding argumentation. It thus becomes interesting to study 6-dimensional Riemannian manifolds carrying the fields Φ, A and H. As the spectrum and interactions of the superstrings are supersymmetric, the effective field theories of superstring theories feature supersymmetry by construction. Therefore, there must be a nowhere-vanishing spinor ϵ that generates the supersymmetry transformations. Note that, as argued above, the *Spin*-structures decompose for direct product ansätze for the background spacetime, and the nowhere-vanishing spinor has to factorize as $\epsilon_{10} = \epsilon_4 \otimes \epsilon_6$. In particular, ϵ_6 is a nowhere-vanishing spinor on M^6, i. e. it is independent of the coordinates of M^4. It spans a one-dimensional subbundle of the spinor bundle on M^6, whose structure group must therefore be reducible to a proper subgroup of $Spin(6)$. This turns out to be $SU(3)$, which then also is the structure group of M^6. We refer to section 2.1 for a mathematical treatment of G-structures.

The task is thus to find 6-dimensional manifolds with $SU(3)$-structure, and then construct solutions to the field equations (1.3.5) of the low-energy effective field theory.
This system simplifies further if we put $H = 0$ and $\Phi = const$. In this situation, the vanishing of the first supersymmetry variation in (1.3.1) requires the existence of a spinor that is parallel with respect to the spin connection induced by the Levi-Civita connection of the metric. Such a parallel spinor is either trivial or nowhere-vanishing. In the latter case this does not only restrict the holonomy group of M^6 to be contained in $SU(3)$, but one can also show that Riemannian manifolds carrying a parallel spinor must be Ricci-flat. This restricts the internal space to be a Calabi-Yau manifold, since these are compact $2n$-dimensional manifolds with holonomy group in $SU(n)$ [6].
Compactification on Calabi-Yau spaces hence yields a set of field equations much simpler than the original one and enables us to use the rich mathematical tools developed within this framework. Compactifications on Calabi-Yau manifolds have been studied extensively since the early days of superstring theories, for instance in [11, 13] and references therein.
Taking a closer look it has been discovered that these models suffer from problems unacceptable from a phenomenological point of view. The reason is that certain geometric degrees of freedom of the internal space are not coupled to the remaining theory. These directions in the space of possible structures on the internal manifold are called moduli. Due to the decoupling of the geometries of M^4 and M^6, they can be chosen independently at every point $x \in M^4$. Therefore, they appear as fields on the effective spacetime with values in the moduli space of the internal manifold. In

Calabi-Yau compactifications, these moduli fields turn out to not have a potential. Therefore, their vacuum expectation values are not under control. However, moduli may determine observable features of the theory, as for example the size of the internal space. This, of course, has to be restricted in order not to spoil consistency with the phenomenological motivation of the compactification procedure.

Letting go of the restriction $\Phi = 0$ and $H = 0$, which lead to compactifications on Calabi-Yau manifolds, we arrive at the subject of flux compactifications. These are more general compactifications allowing for non-trivial H-field strength and dilaton field. These might introduce potentials for the moduli fields via Yukawa couplings and could, therefore, make the vacuum expectation values of the moduli fields take phenomenologically acceptable values. While this works in certain examples (see for instance [13,14] and references therein), so far there has not been found a mechanism which fixes the moduli problem generically.
For the models we consider we keep the simplifying assumptions that led to H, F_A and Φ being non-trivial on the internal space only. Such compactifications of the heterotic string theories have first been investigated in [15].
Still, compactifications do not simply arise from string theory, but one has to choose a geometric ansatz and then investigate the low-energy effective field theory in order to find vacuum solutions. It is hoped that ultimately a mechanism will be found which determines spacetime and its geometry dynamically. However, it seems unlikely that this is possible without considering back-reactions of strings to the background geometry. For example, we first might choose a certain geometry of the internal space M^6, make ansätze for some of the fields and then try to extend this set of fields to a solution of either the heterotic supersymmetry or supergravity equations. The equations for heterotic supersymmetry, or BPS equations, are given by requiring the supersymmetry variations (1.3.1) to vanish identically, while the field equations of heterotic supergravity are (1.3.5). Such extensions have successfully been constructed for example in [12], starting from solutions to the gauge sector and making an ansatz for H.

This is the motivation for this thesis. We will construct 6-dimensional manifolds with $SU(3)$-structure that have a geometry better accessible than that of just a generic $SU(3)$ 6-manifold. Subsequently, we will search for solutions to the so-called instanton equation induced by these $SU(3)$-structures. The instanton equation implies the Yang-Mills equations with a certain torsion term, thus yielding an ansatz for H. Also, it implies the gravitino equation, which is the requirement that the third variation in (1.3.1) vanishes. Furthermore, it has been shown [16] that the heterotic supergravity equations (1.3.1) together with the anomaly cancellation condition (1.3.2) imply the field equations (1.3.5) in quite general situations.
However, in the next three chapters, we will take a detour through the geometry of G-structures at first. We will see how one can classify these, before we specialize to

$SU(2)$-structures in 5 dimensions and $SU(3)$-structures in 6 dimensions. We then investigate the instanton condition induced by a G-structure in general and finally bring full circle this detour by seeing how the spinorial version of the instanton equation arising in (1.3.1) is linked to the more general version of the instanton condition we will encounter in section 4.1.

Chapter 2

G-Structures and Intrinsic Torsion

2.1 Fiber Bundles and G-Structures

In this section, we introduce the notion of a G-structure on a principal bundle. The results obtained here are already known in the literature [17–19], but we choose to provide an exposition of the principles for the sake of completeness.

Gauge theories, spin geometry and G-structures are formulated in the language of principal fiber bundles and associated vector bundles. Their construction and elaboration of their properties can be found in the literature, as for example in [18]. This also is the reference for the short exposition in appendix A, to which we refer the reader for a compact introduction to the formalism and the notation we use throughout this text.

Consider a principal fiber bundle $(\mathcal{P}, \pi, M, H)$ and a representation ρ of H on a k-dimensional $\mathbb{K}$-vector space V. This data gives rise to the associated vector bundle

$$E = \mathcal{P} \times_{(H,\rho)} V. \tag{2.1.1}$$

Assume that E is endowed with a non-vanishing section such that around every $x \in M$ there is a local frame of E with respect to which this section has certain fixed, constant coefficients.
As an example, one may think of a Riemannian metric $g \in \Gamma(\vee^2 T^*M)$ on M. Around every point of M one can find local orthonormal frames for g. Hence, a Riemannian metric is a section in a vector bundle associated to $F(TM)$ having the aforementioned property.
Any section with this property also defines distinguished bases of the fibers, namely it singles out those bases with respect to which it has constant coefficients. Different bases with this property are related by linear transformations that leave these coefficients unchanged. This defines a subgroup G of $GL(k, \mathbb{K})$, namely the stabilizer of the constant coefficients in V. In the example of a Riemannian metric, this group is precisely the group $O(D)$ of orthonormal transformations.
Again one can show (and we will do so below) that the set of these bases defines a principal G-bundle, which is in particular a reduction of the frame bundle. This is called a G-structure.

We will now make this more heuristic exposition mathematically precise. The following defines a G-structure to be a special type of bundle reduction (see A.2 for the definition of a bundle reduction). It is a slightly generalized version of the notion given in [17].

Definition 2.1.1: *Let $(\mathcal{P}, \pi, M, H)$ be a principal H-bundle over M. We define a* ***G-structure on $\mathcal{P}$*** *to be a reduction (λ, ι) of $\mathcal{P}$ to a principal bundle $(\mathcal{Q}, \pi, M, G)$, where ι is the inclusion map.*
For $H = GL(D, \mathbb{R})$, $G \subset H$ we refer to a G-structure on $F(TM)$ as a ***G-structure on M****.*

In fact, one could use more general bundle reductions (e. g. [18]), but for our purposes this will be sufficient. However, this definition puts restrictions on λ.

Lemma 2.1.2: *In the above definition of a G-structure, λ must always be injective.*

Proof. Assume there are $g, g' \in G$ such that $g \neq g'$ and $\lambda(g) = \lambda(g')$. Then, for $q \in \mathcal{Q}$, $R^{\mathcal{P}}_{\lambda(g)}\iota(q) = R^{\mathcal{P}}_{\lambda(g')}\iota(q)$ and equivalently, $\iota(q) = R^{\mathcal{P}}_{\lambda(g'\,g^{-1})}\iota(q)$. But, since ι is injective, this is true if and only if $q = R^{\mathcal{Q}}_{g'\,g^{-1}}q$. As $R^{\mathcal{Q}}$ is simply transitive, this implies $g = g'$. □

Thus, the Lie group homomorphisms we can use in the construction of G-structures due to definition 2.1.1 are injective. As $\iota(\mathcal{Q}) \subset \mathcal{P}$ is an embedded submanifold that is closed under the right-action of G given by $g \mapsto R^{\mathcal{P}}_g$, the λ are *embeddings* of G into H. Therefore, the following definition is equivalent to definition 2.1.1.

Definition 2.1.3: *Let $(\mathcal{P}, \pi, M, H)$ be a principal H-bundle. A* ***G-structure on $\mathcal{P}$*** *is a principal* subbundle *$(\mathcal{Q}, \pi, M, \lambda(G))$ of $\mathcal{P}$ having structure group $\lambda(G)$ for some embedding $\lambda : G \hookrightarrow H$.*

We will make use of definition 2.1.3 in the following, and we will most of the time identify G with its embedding $\lambda(G)$. Thus, we drop the explicit λ and view the structure group of the G-structure as a *subgroup* of $GL(D, \mathbb{R})$.

In the heuristic approach above, we used a certain section of an associated vector bundle to define a principal subbundle of $F(E)$ and thus a G-structure. A Riemannian metric, for instance, defines a unique $O(D)$-structure on M by singling out the subbundle of $F(TM)$ consisting of orthonormal frames. We may now ask whether there is a general link between sections of the above kind and G-structures on a principal bundle.
First, we need the statement that bundles associated to $\mathcal{P}$ may also be constructed from the reduced principal bundle of a G-structure.

Lemma 2.1.4: *Assume that the principal bundle* $(\mathcal{P}, \pi, M, H)$ *admits a* G*-structure* $(\mathcal{Q}, \pi, M, G)$*, and that* $E = \mathcal{P} \times_{(H,\rho)} V$ *is an associated vector bundle for* $\mathcal{P}$ *and the* H*-representation* ρ *on* V*.*
Then we have

$$E = \mathcal{P} \times_{(H,\rho)} V = \mathcal{Q} \times_{(G,\rho_{|G})} V. \tag{2.1.2}$$

Proof. E consists of equivalence classes $[p, v]$, $p \in \mathcal{P}, v \in V$, where

$$[p, v] = [R_h\, p, \rho(h^{-1})(v)] \;\forall h \in H. \tag{2.1.3}$$

By the transitivity of the H-action on the fibers of $\mathcal{P}$, to every $p \in \mathcal{P}$ we can find an $h \in H$ such that $R_h\, p = q \in \mathcal{Q}$. That is, every $[p, v] \in \mathcal{P} \times_{(H,\rho)} V$ has a representative $[q, \rho(h^{-1})(v)]$ where $q \in \mathcal{Q}$.
This, however, is the very definition of an element of $\mathcal{Q} \times_{(G,\rho_{|G})} V$, due to the fact that $\mathcal{Q} \subset \mathcal{P}$ is a principal subbundle.
By the same reasoning, every $[q, v] \in \mathcal{Q} \times_{(G,\rho_{|G})} V$ is already an element of the associated bundle $\mathcal{P} \times_{(H,\rho)} V$. □

Note that it is crucial that the bundles are exactly identical and not just isomorphic, as they would be if we had used more general bundle reductions instead of embedded subbundles. Perhaps the most common example of this assertion is that the tangent bundle TM of a semi-Riemannian manifold can be built from either any local frames or just the orthonormal frames, i. e.

$$TM = F(TM) \times_{(GL(D,\mathbb{R}),\rho)} \mathbb{R}^D = SO(M, g) \times_{(SO(p,q),\rho)} \mathbb{R}^D. \tag{2.1.4}$$

Here, $SO(M, g)$ is the bundle of orthonormal frames defined by g, and ρ is the standard representation of $GL(D, \mathbb{R})$ on $\mathbb{R}^D$.

Of course, ρ can be any H-representation on V. In particular, it is not required to be irreducible. Furthermore, even if ρ is irreducible as an H-representation, its restriction to G as a G-representation on V will in general be reducible. In this case, the G-representation decomposes into irreducible representations (at least for G being compact). The induced decomposition of $V = \bigoplus_{i=1}^{N} V_i$ into invariant subspaces V_i furnishes a decomposition of E.

Definition 2.1.5: *We call a vector bundle* $E = \mathcal{P} \times_{(H,\rho)} V$ ***irreducible****, if the representation* ρ *of* H *on* V *is irreducible.*

Note that the property of irreducibility depends on the structure group principal bundle, that E is associated to. Thus, we have

Proposition 2.1.6: *Consider $(\mathcal{P}, \pi, M, H)$ a principal H-bundle over M admitting a G-structure $\mathcal{Q}$, where G is compact subgroup of H. Let ρ be a representation of H on the vector space V.*
Then $\rho_{|G}$ will in general be reducible to irreducible representations ρ_i, and there is a decomposition of $E = \mathcal{P} \times_{(H,\rho)} V$:

$$E = \bigoplus_{i=1}^{N} \mathcal{Q} \times_{(G,\rho_i)} V_i = \bigoplus_{i=1}^{N} E_i, \tag{2.1.5}$$

where each of the E_i, $i = 1, \ldots, N$ is irreducible as a vector bundle associated to $\mathcal{Q}$.

This yields

Corollary 2.1.7: *If one of the terms in the splitting (2.1.5) is a one-dimensional vector bundle associated to $\mathcal{Q}$ via the trivial representation of G, then this vector bundle allows for a nowhere-vanishing section. Furthermore, there is a section which has fixed local representation with respect to any local section s of $\mathcal{Q}$. Every such section of a vector bundle is parallel with respect to every connection on $\mathcal{Q}$.*

Proof. Define a section $e \in \Gamma(E_{i_0})$ via

$$e(\pi(q)) \coloneqq [q, v_{i_0}] \tag{2.1.6}$$

for a fixed $v_{i_0} \in V_{i_0}$. Because ρ_{i_0} is trivial, this is independent of the choice of the point of the fiber. Therefore, this yields a globally well-defined, nowhere-vanishing section. In particular, its local representation is given by v_{i_0} with respect to any $q \in \mathcal{Q}$, and, thus, also with respect to any local section of $\mathcal{Q}$.
From this we see that the pendant of e on $\mathcal{Q}$ is a constant map $v_{i_0} : \mathcal{Q} \to V_{i_0}$, $q \mapsto v_{i_0}$.
Additionally, $\rho_{i_0 *} = 0$, whence $D_A v_{i_0} = 0 \; \forall\, A \in \mathcal{C}(\mathcal{Q})$. □

Equivalently, one could choose an arbitrary non-zero element of $E_{i_0|x}$ for any $x \in M$ and use parallel transport with respect to an arbitrary connection on $\mathcal{Q}$ to extend this to a section of E_{i_0}. In particular, in this case E admits a nowhere-vanishing section. Since these particular sections are parallel with respect to every connection on $\mathcal{Q}$, one often calls them *invariant sections.*
Thus, we have seen that the existence of G-structures on $\mathcal{P}$ implies the existence of nowhere-vanishing sections on certain associated vector bundles. The only criterion for the existence of such distinguished sections is how the H-representation decomposes into irreducible G-representations.

It will be important for us to also investigate the converse direction here. A nowhere-vanishing section of an irreducible associated vector bundle of rank k defines a splitting of E into a rank-$(k-1)$ and a rank-1 subbundle. Hence, one could expect that under certain circumstances these subbundles are associated vector bundles themselves. But if they are, the structure group has to be reduced since E was irreducible as associated to $\mathcal{P}$, hence implying the existence of a G-structure on $\mathcal{P}$. The result is stated in

Proposition 2.1.8: *Let $(\mathcal{P}, \pi, M, H)$ be a principal bundle over M, and consider a $\mathbb{K}$-vector bundle $E = \mathcal{P} \times_{(H,\rho)} V$ associated to $\mathcal{P}$. Let there be an element $\tau_0 \in V$ having a non-trivial stabilizer $G = \{h \in H \,|\, \rho(h)(\tau_0) = \tau_0\}$.*
In this case, E admits a nowhere-vanishing section $\tau \in \Gamma(E)$ such that there is a covering $\{(U_i, s_i)\}_{i\in\Lambda}$ of M by local sections of $\mathcal{P}$ satisfying

$$\tau_{|x} = [s_i(x), \tau_0] \quad \forall\, x \in U_i, i \in \Lambda, \tag{2.1.7}$$

if and only if there exists a G-structure on $\mathcal{P}$.

Proof. Assume we are given a section $\tau \in \Gamma(E)$ and a covering $\{(U_i, s_i)\}_{i\in\Lambda}$ with the described property. First, consider $U_i \cap U_j =: U_{ij} \neq \emptyset$. On U_{ij} we have

$$\tau_{|x} = [s_i(x), \tau_0] = [s_j(x), \tau_0] \quad \forall\, x \in U_{ij}. \tag{2.1.8}$$

There exists a smooth transition map $h_{ij} : U_{ij} \to H$, $x \mapsto h_{ij}(x)$ such that

$$s_i(x) = R_{h_{ij}(x)}\, s_j(x) \quad \forall\, x \in U_{ij}. \tag{2.1.9}$$

Thus,

$$\begin{aligned} \tau_{|x} &= [s_i(x), \tau_0] = [R_{h_{ij}(x)}\, s_j(x), \tau_0] \\ &= [s_j(x), \rho(h_{ij}(x))(\tau_0)] \\ &= [s_j(x), \tau_0]. \end{aligned} \tag{2.1.10}$$

Since the map $v \mapsto [p, v]$ for fixed $p \in \mathcal{P}$ is a diffeomorphism (appendix A.1), h_{ij} takes values in G exclusively.
Now define $\mathcal{Q} = \{p \in \mathcal{P} \,|\, \tau_{|\pi(p)} = [p, \tau_0]\}$. We will show that this set defines the required G-structure.
For $i \in \Lambda$ consider the maps

$$\phi_i : \mathcal{Q}_{|U_i} \to U_i \times G, \;\; \phi_i(R_g\, s_i(x)) := (x, g). \tag{2.1.11}$$

From the above arguments we know that transitions between these maps are given by smooth G-cocycles. Thus, we endow $\mathcal{Q}$ with the differentiable structure induced by the local trivializations which are given by the ϕ_i. We see that this is in fact a principal G-bundle.
The ϕ_i directly extend to local trivializations of $\mathcal{P}$ by $\phi_i(R_h\, s_i(x)) := (x, h)$. These are compatible with the differentiable structure of $\mathcal{P}$ in the sense that this bundle

atlas of $\mathcal{P}$ *restricts* to a bundle atlas of $\mathcal{Q}$. We, therefore, identify $\mathcal{Q}$ as a principal G-subbundle of $\mathcal{P}$ and have thus constructed a G-structure $(\mathcal{Q}, \pi, M, G)$ on $\mathcal{P}$.

If, on the other hand, there is a G-structure $\mathcal{Q} \subset \mathcal{P}$ on $\mathcal{P}$, consider a covering $\{(U_i, s_i)\}_{i\in\Lambda}$ of M with local sections of $\mathcal{Q}$. As in the preceding corollary, define $\tau_{|x} = [q, \tau_0] \; \forall \, x \in M$. We saw in the proof of that corollary that this defines a global section of E. Moreover, $\tau_{|x} = [s_i(x), \tau_0] \; \forall \, i \in \Lambda, \; x \in U_i$. □

This proposition shows how the choice of nowhere-vanishing sections with fixed local representations defines G-structures on principal bundles and vice versa. Note that often these sections sections arise in tuples. This is because generically there will be several representations of H whose restrictions to $G \subset H$ contain a trivial representation. Each of these will then give rise to a nowhere-vanishing section of the corresponding associated vector bundle. However, for the sake of simplicity we chose to treat only one of these sections in the above considerations. Let us finally give these distinguished sections a name for further reference.

Definition 2.1.9: *Be $(\mathcal{P}, \pi, M, H)$ a principal fiber bundle over M and $(\mathcal{Q}, \pi, M, G)$ a G-structure on $\mathcal{P}$ which is defined by a section $\tau \in \Gamma(E)$ in the fashion of proposition 2.1.8. Then τ is called a* ***defining section*** *for $\mathcal{Q}$.*

Reductions of $(\mathcal{P}, \pi, M, H)$ to principal H-bundles can equivalently be defined via global sections of the associated bundle $\mathcal{P} \times_{(\ell,H)} H/G$ [18], where the left-action $\ell : H \times H \to H$, $\ell(h_1)(h_2) = h_1 h_2$ is the natural left-action of H on itself. However, we considered it more valuable to investigate G-structures by means of defining sections in the above way. This view is more directly related with calculations in later chapters, where we benefit from using the components of the defining sections with respect to local frames on M.

To sum up, the existence of G-structures of a principal H-bundle, where G is the stabilizer of some element in a representation ρ of H on a vector space, is equivalent to the existence of a section of $E = \mathcal{P} \times_{(H,\rho)} V$ having property (2.1.7).

Returning to the example of a Riemannian metric, $O(D) \subset GL(D, \mathbb{R})$ is a closed subgroup defined as the stabilizer of a certain element of $\vee^2(\mathbb{R}^D)^*$. A Riemannian metric on M does, of course, satisfy the assumptions of proposition 2.1.8. This is because everywhere there exists a local orthonormal frame for any given g. The subbundle $\mathcal{Q}$, as constructed in the proof of this proposition, is just the subbundle of orthonormal frames as contained in $F(TM)$. Thus, our formal approach applied to the motivating example of this section reproduces what we used as a guiding principle.

In the following we will mostly deal with G-structures on M, instead of on a more general principal bundle $\mathcal{P}$.

2.2 Intrinsic Torsion and Classification of G-Structures

We now turn to the study of connections on principal bundles carrying a G-structure as defined in the previous section. The central issue here is that, generically, connections on $\mathcal{P}$ do not restrict to connections on the principal bundle of the G-structure $\mathcal{Q}$. In particular, when working with G-structures on M it is not necessarily true that the Levi-Civita connection of a Riemannian metric compatible with the G-structure has holonomy contained in G. That is, if $\mathcal{Q}$ is given by defining sections, in general the Levi-Civita connection will not be compatible with the G-structure in the sense that it fails to preserves the defining sections.
For example, the five-sphere S^5 with its standard metric carries an $SU(2)$-structure, but has a bigger holonomy group. The former will be shown in section 3.4 using $S^5 = SU(3)/SU(2)$.

Being given a connection A on $\mathcal{Q}$, there always exists an extension of A to a connection $A_{\mathcal{P}}$ on $\mathcal{P}$ (cf. appendix A.2, [18]). The restriction to $\mathcal{Q}$ then yields back $A \in \mathcal{C}(\mathcal{Q})$. Thus, there is an embedding

$$\mathcal{C}(\mathcal{Q}) \hookrightarrow \mathcal{C}(\mathcal{P}). \tag{2.2.1}$$

This yields

Proposition 2.2.1: *Consider a principal H-bundle $(\mathcal{P}, \pi, M, H)$ admitting a G-structure $(\mathcal{Q}, \pi, M, G)$. The following two assertions hold true:*

(1) Every connection on $\mathcal{Q}$ can be extended to a connection on $\mathcal{P}$.

(2) The restriction of a connection 1-form on $\mathcal{P}$ to $\mathcal{Q}$ is a connection 1-form on $\mathcal{Q}$ if and only if there is a covering $\{(U_i, s_i)\}_{i\in\Lambda}$ of M with local sections of $\mathcal{Q}$ such that $s_i^* A$ is $\mathfrak{g}$-valued for every $i \in \Lambda$.

Proof. We are left to show the second assertion. Consider a covering $\{(U_i, s_i)\}_{i\in\Lambda}$ of M with local sections of $\mathcal{Q}$, and let $A \in \mathcal{C}(\mathcal{P})$ be a connection on $\mathcal{P}$. We compute

$$s_j^* A = (R_{g_{ji}} \circ s_i)^* A = Ad_H(g_{ji}^{-1}) \circ s_i^* A + {g_{ji}}^* \mu_G, \tag{2.2.2}$$

where g_{ji} is the transition map from s_i to s_j. Hence, the collection $(s_i^* A)_{i\in\Lambda}$ has the correct transformation behavior of a local representation of a connection on $\mathcal{Q}$. However, as A is a generic connection on $\mathcal{P}$, which has structure group H, $s_i^* A$ will in general be $\mathfrak{h}$-valued, rather than $\mathfrak{g}$-valued. Therefore, the collection $(s_i^* A)_{i\in\Lambda}$ represents a connection on $\mathcal{Q}$ if and only if $s_i^* A$ takes values in $\mathfrak{g}$ for all $i \in \Lambda$. □

As simple illustration of the second statement is that while every metric connection on a Riemannian manifold (M, g) is a connection on $F(TM)$, not every connection

on $F(TM)$ preserves g.
While this already states a relation between connections on $\mathcal{P}$ and $\mathcal{Q}$, for certain Lie groups we can construct connections on $\mathcal{Q}$ from general connections on $\mathcal{P}$. This will be of importance in the following sections when we consider *metric G-structures* on M. These we define to be G-structures on M where $G \subset SO(D) \subset GL(D, \mathbb{R})$. That is, $\mathcal{Q}$ is contained in the bundle of orthonormal frames that is defined by a certain Riemannian metric on an orientable M.

Proposition 2.2.2: *Let $(\mathcal{P}, \pi, M, H)$ a principal H-bundle admitting a G-structure $(\mathcal{Q}, \pi, M, G)$ where the splitting $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ is such that m is invariant under the action of G via Ad_H. Denote by $pr_{\mathfrak{g}}$ and pr_{m} the projections of $\mathfrak{h}$ onto $\mathfrak{g}$ and m, respectively, and let $A \in \mathcal{C}(\mathcal{P})$ be an arbitrary connection on $\mathcal{P}$.*
Then

$$p_{\mathcal{Q}}(A) \coloneqq pr_{\mathfrak{g}} \circ A_{|\mathcal{Q}} \in \mathcal{C}(\mathcal{Q}) \tag{2.2.3}$$

is a connection on $\mathcal{Q}$. Furthermore, there is a horizontal m-valued 1-form of type Ad_H on $\mathcal{Q}$ given by

$$T_{\mathcal{Q}}(A) \coloneqq pr_{\mathrm{m}} \circ A_{|\mathcal{Q}} = A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A) \in \Omega^1_{hor}(\mathcal{Q}, \mathrm{m})^{(G, Ad_H)}. \tag{2.2.4}$$

Proof. First, we show that $p_{\mathcal{Q}}(A)$ is a connection 1-form on $\mathcal{Q}$.
Be $\xi \in \mathfrak{g}$. The fundamental vector field on $\mathcal{Q}$ corresponding to ξ is given by

$$\varphi^{\mathcal{Q}}(\xi)_{|q} = \frac{\mathrm{d}}{\mathrm{d}t}_{|0} R^{\mathcal{Q}}_{\exp(t\xi)}\, q = \frac{\mathrm{d}}{\mathrm{d}t}_{|0} R^{\mathcal{P}}_{\exp(t\xi)}\, q = \varphi^{\mathcal{P}}(\xi)_{|\mathcal{Q}} \quad \forall\, q \in \mathcal{Q}. \tag{2.2.5}$$

The second equality holds because $\mathcal{Q}$ is a principal subbundle of $\mathcal{P}$. Hence, we have

$$\varphi^{\mathcal{Q}}(\xi) = \varphi^{\mathcal{P}}(\xi)_{|\mathcal{Q}} \quad \forall\, \xi \in \mathfrak{g}. \tag{2.2.6}$$

Therefore,

$$p_{\mathcal{Q}}(A)_q \circ \varphi^{\mathcal{Q}}(\xi) = pr_{\mathfrak{g}} \circ A_q \circ \varphi^{\mathcal{P}}(\xi) = pr_{\mathfrak{g}}(\xi) = \xi \;\; \forall\, \xi \in \mathfrak{g},\, q \in \mathcal{Q}, \tag{2.2.7}$$

and $p_{\mathcal{Q}}(A) \circ \varphi^{\mathcal{Q}} = id_{\mathfrak{g}}$ is satisfied.
Now consider

$$\begin{aligned} R_g{}^*\big(p_{\mathcal{Q}}(A)\big) &= pr_{\mathfrak{g}} \circ A \circ R_{g*} \\ &= pr_{\mathfrak{g}} \circ R_g{}^* A \\ &= pr_{\mathfrak{g}} \circ Ad_H(g^{-1}) \circ A \\ &= Ad_H(g^{-1}) \circ pr_{\mathfrak{g}} \circ A \\ &= Ad_G(g^{-1}) \circ p_{\mathcal{Q}}(A). \end{aligned} \tag{2.2.8}$$

Here we used that, as representations of G on $\mathfrak{g}$, the restriction of the adjoint representation of H to G coincides with the adjoint representation of G. That is, we have

$Ad_{H\,|G} = Ad_G$ as representations of G on $\mathfrak{g}$. This is because $Ad_H(g) = \alpha_H(g)_{*|e}$, where the map $\alpha_H(h)(a) = h\,a\,h^{-1}$ is the inner automorphism of H. The group multiplication on G and the restriction of that on H to elements of G coincide, whence $Ad_H(g)_{|\mathfrak{g}} = Ad_G(g)\ \forall\, g \in G$. Also, the fourth equality does only hold true for the splitting $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ being invariant under the restriction of Ad_H to G. For m being invariant under the action of G via Ad_H, the coincidence of $Ad_H(g)$ and $Ad_G(g)$ on $\mathfrak{g}$ then yields the invariance of the splitting of $\mathfrak{h}$ and thereby the fifth equality.
Thus, $p_{\mathcal{Q}}(A)$ is a connection form on $\mathcal{Q}$, i. e. $p_{\mathcal{Q}}(A) \in \mathcal{C}(\mathcal{Q})$.

Let us turn to the proof of equation (2.2.4). Since $p_{\mathcal{Q}}(A) = A_{|\mathcal{Q}} - (A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A))$ takes values in $\mathfrak{g}$, the part of $A_{|\mathcal{Q}}$ mapping to m is precisely given by $A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A)$, which hence is an m-valued 1-form on $\mathcal{Q}$.
Also, as shown in the proof of (2.2.3), $A_{|\mathcal{Q}}$ and $p_{\mathcal{Q}}(A)$ coincide on the fundamental vector fields on $\mathcal{Q}$ and hence on the complete vertical tangent bundle of $\mathcal{Q}$. Consequently, $A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A)$ is horizontal.
We are left to check that this form is of type Ad_H, which is only well-defined on m-valued forms on $\mathcal{Q}$ if m is invariant under Ad_H restricted to G. From computing

$$\begin{aligned} R_g^*(A_{|R_g q} - p_{\mathcal{Q}}(A)_{|R_g q}) &= {R_g}^*(A_{|R_g q}) - {R_g}^*\big(p_{\mathcal{Q}}(A)_{|R_g q}\big) \\ &= Ad_H(g^{-1}) \circ A_{|q} - pr_{\mathfrak{g}} \circ Ad_H(g^{-1}) \circ A_{|q} \\ &= Ad_H(g^{-1}) \circ (A_{|q} - pr_{\mathfrak{g}} \circ A_{|q}) \end{aligned} \tag{2.2.9}$$

we see that $A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A)$ does indeed have this property, thus completing the proof. Note that in the third equality we have again made use of the invariance of the splitting of $\mathfrak{h}$. □

The first assertion of the proposition has already been proven for example in [18]. For G being a connected Lie group, the invariance property is satisfied if and only if the splitting $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ is reductive, that is, $[\mathfrak{g}, \mathrm{m}]_H \subset \mathrm{m}$. This is true since for any Lie group homomorphism $\phi : G \to H$ one can show that $\phi \circ \exp_G = \exp_H \circ \phi_*$ (cf. [18]), and since every element of the identity component of a Lie group can be expressed as a product of elements of any arbitrary open neighborhood of the identity.
The m-valued 1-form $T_{\mathcal{Q}}(A) \in \Omega^1_{hor}(\mathcal{Q}, \mathrm{m})^{(G, Ad_H)}$ has a remarkable property.

Lemma 2.2.3: *Consider a principal H-bundle $(\mathcal{P}, \pi, M, H)$ admitting a G-structure $\mathcal{Q}$, where the splitting $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ is invariant under the action of Ad_H restricted to G. For any $A \in \mathcal{C}(\mathcal{P})$ we have*

$$pr_{\mathrm{m}} \circ (A_{|\mathcal{Q}} - A') = T_{\mathcal{Q}}(A) \quad \forall\, A' \in \mathcal{C}(\mathcal{Q}). \tag{2.2.10}$$

Thus, for fixed $A \in \mathcal{C}(\mathcal{P})$, $T(A)$ defines a map

$$T(A) : G\mathcal{P} \to \Omega^1_{hor}(\mathcal{Q}, \mathrm{m})^{(G, Ad_H)}, \ \ \mathcal{Q} \mapsto T_{\mathcal{Q}}(A) \tag{2.2.11}$$

from the space $G\mathcal{P}$ of G-structures on $\mathcal{P}$ to $\Omega^1_{hor}(\mathcal{Q}, \mathrm{m})^{(G,Ad_H)}$.

Proof. $\mathcal{C}(\mathcal{Q})$ is an affine vector space over $\Omega^1_{hor}(\mathcal{Q}, \mathfrak{g})^{(G,Ad_G)}$. Recall from the previous proposition that $p_{\mathcal{Q}}(A) \in \mathcal{C}(\mathcal{Q})$. From this we infer

$$p_{\mathcal{Q}}(A) - A' \in \Omega^1_{hor}(\mathcal{Q}, \mathfrak{g})^{(G,Ad_G)}. \tag{2.2.12}$$

Thus,

$$\begin{aligned} pr_m \circ (A_{|\mathcal{Q}} - A') &= pr_m \circ \big((A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A)) + \underbrace{(p_{\mathcal{Q}}(A) - A')}_{\mathfrak{g}\text{-valued}}\big) \\ &= pr_m \circ (A_{|\mathcal{Q}} - p_{\mathcal{Q}}(A)) \\ &= T_{\mathcal{Q}}(A), \end{aligned} \tag{2.2.13}$$

and the proof is complete. □

Because $T_{\mathcal{Q}}(A)$ depends on the choice of a connection on $\mathcal{P}$ and on the G-structure on $\mathcal{P}$ only, we can for a fixed $A_0 \in \mathcal{C}(\mathcal{P})$ use $T_{\mathcal{Q}}(A_0)$ in order to classify G-structures on $\mathcal{P}$.

Definition 2.2.4: *On a principal H-bundle $(\mathcal{P}, \pi, M, H)$ admitting a G-structure $\mathcal{Q}$, where the splitting $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ is invariant under the action of Ad_H restricted to G, fix a connection $A_0 \in \mathcal{C}(\mathcal{P})$.*
Then $T_{\mathcal{Q}}(A_0) \in \Omega^1_{hor}(\mathcal{Q}, \mathrm{m})^{(G,Ad_H)}$ is called the ***intrinsic torsion of $\mathcal{Q}$ with respect to*** *A_0.*

Later on in this thesis, we will mostly focus on metric G-structures on orientable Riemannian manifolds (M, g). On the bundle of orthonormal frames $SO(M, g)$ of such spaces we are given a distinguished connection, namely the Levi-Civita connection of g. In this case, we choose A_0 to be the Levi-Civita connection and refer to $T_{\mathcal{Q}}(A_0)$ as the *intrinsic torsion of the G-structure on M*. This way we recover the notion of intrinsic torsion used in [20], which will be sufficient for our purposes. In fact, there is a more general way to define intrinsic torsion on arbitrary frame bundles used e. g. in [19, 21]. In [19], the link to our elaboration is given precisely by measuring intrinsic torsion of metric G-structures with respect to the Levi-Civita connection of the given metric.
In particular, the torsion $T_{\mathcal{Q}}(A_0)$ measures how much A_0 fails to be a connection on $\mathcal{Q}$, i. e. how much it fails to preserve the (defining sections of the) G-structure.

Definition 2.2.5: *If $T_{\mathcal{Q}}(A_0) = 0$, we say that the G-structure $\mathcal{Q}$ is* ***integrable with respect to*** *A_0. If $T_{\mathcal{Q}}(A^g) = 0$ in the case of metric G-structures on M, where we* *choose $A_0 = A^g$ to be the Levi-Civita connection of the metric compatible with $\mathcal{Q}$, we just call $\mathcal{Q}$* ***integrable****.*

For the splitting $\mathfrak{h} = \mathfrak{g} \oplus m$ being invariant under the G-representation Ad_H, we have (again in the notation of [18])

$$\begin{aligned} Ad(\mathcal{P}) &\coloneqq \mathcal{P} \times_{(H,Ad_H)} \mathfrak{h} \\ &= \mathcal{P} \times_{(H,Ad_H)} (\mathfrak{g} \oplus \mathrm{m}) \\ &= Ad(\mathcal{Q}) \oplus \mathcal{Q} \times_{(G,Ad_H)} \mathrm{m} \\ &= Ad(\mathcal{Q}) \oplus \mathcal{V}. \end{aligned} \tag{2.2.14}$$

Here we used lemma 2.1.4 and again the fact that Ad_H restricted to G coincides with Ad_G as a representation of G on $\mathfrak{g}$.
In [18] it is shown that on every principal H-bundle $\mathcal{P}$ there exists a bijective map $\Psi : \Omega^k_{hor}(\mathcal{P}, V)^{(G,\rho)} \to \Omega^k(M, \mathcal{P} \times_{(G,\rho)} V)$ for any $k \in \mathbb{N}_0$ (see appendix A.1 for more details). Thus, we see from (2.2.4) that $\Psi(T_{\mathcal{Q}}(A_0)) \in \Omega^1(M, \mathcal{V}) = \Gamma(T^*M \otimes \mathcal{V})$. If, in particular, we are dealing with G-structures on $F(TM)$, rather than on some general principal bundle, T^*M is an associated bundle of $\mathcal{Q}$ as well. Hence, so is $T^*M \otimes \mathcal{V}$, but it may be reducible as associated to $\mathcal{Q}$. In general, we thus have

$$\begin{aligned} T^*M \otimes Ad(\mathcal{P}) &= T^*M \otimes (Ad(\mathcal{Q}) \oplus \mathcal{Q} \times_{(G,Ad_H)} \mathrm{m}) \\ &= \big(T^*M \otimes Ad(\mathcal{Q})\big) \oplus \big(\mathcal{Q} \times_{(G,\rho^{-T} \otimes Ad_H)} (\mathbb{R}^D \otimes \mathrm{m})\big) \\ &= \big(T^*M \otimes Ad(\mathcal{Q})\big) \oplus \bigoplus_{i=1}^{N} \mathcal{W}_i. \end{aligned} \tag{2.2.15}$$

We take ρ to denote the standard representation of $GL(D, \mathbb{R})$ on $\mathbb{R}$, and TM is associated to $F(TM)$ via this representation. The cotangent bundle is dual to T^*M, whence it is associated to $F(TM)$ via the inverse transposed representation ρ^{-T}.
The method for classifying G-structures on M is then to state in which of the associated vector bundles $\mathcal{W}_i$, $i \in \{1, \dots, N\}$ the intrinsic torsion $\Psi(T_{\mathcal{Q}}(A_0))$ of the G-structure has non-trivial components.
This idea has been developed in [5], and it has been investigated for metric G-structures on Riemannian manifolds for example in [20] and as well in [22] and [14], where in the latter two references applications to flux compactifications are pointed out.

Chapter 3

Special Geometric Structures

3.1 SU(3)-Structures in D = 6

In the last chapter we explored the general framework behind G-structures. Here we specialize to $SU(3)$-structures and $SU(2)$-structures on 6- and 5-dimensional manifolds, respectively, for we will use these in the constructions of chapters 5 and 6.

Let us begin our introduction of $SU(3)$-structures by first considering a little more general structures on even-dimensional manifolds. These will provide us with background useful in later considerations and computations. The notions used here can be found in e. g. [17, 23].
The most important objects in this section will be the complexified tangent and cotangent bundle

$$TM^{\mathbb{C}} \coloneqq TM \otimes \mathbb{C} \quad \text{and} \quad T^*M^{\mathbb{C}} \coloneqq T^*M \otimes \mathbb{C}. \tag{3.1.1}$$

If $\{e_i \,|\, i = 1, \ldots, D\}$ is a local frame of TM with dual coframe β, we locally have

$$T_U M^{\mathbb{C}} = \text{span}_{\mathbb{C}}\{e_i \,|\, i = 1, \ldots, D\}, \quad T_U^* M^{\mathbb{C}} = \text{span}_{\mathbb{C}}\{\beta_i \,|\, i = 1, \ldots, D\}. \tag{3.1.2}$$

On every complex vector space of even dimension there exist complex structures, i. e. automorphism of the vector space that square to minus the identity. Consequently, if D is even, which shall be assumed for the remainder of this section, we could construct local complex structures from local frames as given above. As it always is with non-trivial vector bundles, there may, however, exist no global versions of these.

Definition 3.1.1: *A manifold endowed with a section $J \in \Gamma(End(TM))$ such that $J^2 = -\mathbb{I}$ is called an* ***almost complex manifold****. J is called an* ***almost complex structure*** *on M. It is the defining section of a $GL(n, \mathbb{C})$-structure on M for $D = 2n$ in the language of section 2.1.*

Such an almost complex structure can be extended linearly to $TM^{\mathbb{C}}$. Since it squares to the identity, $J_{|x}$ has eigenvalues $\pm i$.
By putting

$$(J^*(\mu))(X) \coloneqq \mu(J(X)) \quad \forall\, \mu \in \Omega^1(M),\, X \in \Gamma(TM) \tag{3.1.3}$$

we obtain $J^* \in \Gamma(End(T^*M))$. Its extension to $T^*M^{\mathbb{C}}$ also squares to the identity and therefore has eigenvalues $\pm i$.

This induces a splitting

$$T^*M^{\mathbb{C}} = \mathrm{Eig}_{J^*}(i) \oplus \mathrm{Eig}_{J^*}(-i) =: \Lambda^{1,0}\, T^*M \oplus \Lambda^{0,1}\, T^*M. \tag{3.1.4}$$

One may then define $\Lambda^{p,0}\, T^*M$ and $\Lambda^{0,p}\, T^*M$ as the p-th exterior power of $\Lambda^{1,0}\, T^*M$ and $\Lambda^{0,1}\, T^*M$, respectively. Furthermore, we put

$$\Lambda^{p,q}\, T^*M \coloneqq \big(\Lambda^{p,0}\, T^*M\big) \wedge \big(\Lambda^{0,q}\, T^*M\big), \tag{3.1.5}$$

thus introducing the decomposition

$$\Lambda^k T^*M^{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q}\, T^*M \quad \forall\, k = 0, \ldots, D. \tag{3.1.6}$$

This suggests the notation $\Omega^{p,q}(M) = \Gamma(\Lambda^{p,q}\, T^*M)$.
However, the existence of an almost complex structure is a weaker condition than the existence of a holomorphic atlas of M, i. e. a proper complex structure. This is taken account for in the following definition which includes statements to be found e.g. in [23].

Definition 3.1.2: *An almost complex manifold (M, J) is called a **complex manifold** if either one of the following equivalent statements holds true:*

(1) J is integrable. That is, its Nijenhuis tensor of J vanishes identically, i. e. for all $X, Y \in \Gamma(TM)$

$$N_J(X,Y) \coloneqq [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X, Y] = 0. \tag{3.1.7}$$

(2) The exterior differential satisfies

$$\mathrm{d}\Omega^{1,0}(M) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M). \tag{3.1.8}$$

(3) The exterior differential satisfies

$$\mathrm{d}\Omega^{0,1}(M) \subset \Omega^{1,1}(M) \oplus \Omega^{0,2}(M). \tag{3.1.9}$$

(4) The exterior differential satisfies

$$\mathrm{d}\Omega^{p,q}(M) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M). \tag{3.1.10}$$

(5) There exists an atlas for M with holomorphic transition maps.

In a gravitational theory, of course, an additional field is present, namely a metric on M. Between this and an almost complex structure there can be a fruitful interplay if the fields are compatible in the following way:

Definition 3.1.3: *A Riemannian manifold (M, g) endowed with an almost complex structure J is said to be an* ***almost Hermitian manifold*** *if g is invariant under J, i. e. if*

$$g\big(J(\cdot), J(\cdot)\big) = g(\cdot, \cdot). \tag{3.1.11}$$

A tuple (g, J) compatible in this sense defines a $U(n)$-structure on M [17]. An almost Hermitian manifold is called a ***Hermitian manifold*** *if its underlying almost complex structure is complex.*

For computations it will be useful to find local bases of the eigenbundles of J^*. Be (M, g, J) an almost Hermitian manifold. For a local orthonormal coframe β on M^D with $D = 2n$ define

$$\mu^j \coloneqq \beta^{2j-1},\ \sigma^j \coloneqq \beta^{2j},\ j = 1, \dots, n. \tag{3.1.12}$$

We choose the coframe in a way such that

$$\begin{aligned} J^*(\mu^j) = &-\sigma^j,\ J^*(\sigma^j) = \mu^j,\ \text{or} \\ J(e_{2j-1}) = &\ e_{2j},\ J(e_{2j}) = -e_{2j-1}. \end{aligned} \tag{3.1.13}$$

Note that for the so-called *Kähler form* we have

$$\omega \coloneqq g(J(\cdot), \cdot) = \sum_{i=1}^{n} \beta^{2j-1} \wedge \beta^{2j} = \sum_{i=1}^{n} \mu^j \wedge \sigma^j. \tag{3.1.14}$$

The forms $\{\mu^j \pm i\,\sigma^j \,|\, j = 1, \dots, n\}$ provide a local basis of $T^*M^{\mathbb{C}}$ satisfying

$$J^*(\mu^j \pm i\,\sigma^j) = \pm i\,(\mu^j \pm \sigma^j). \tag{3.1.15}$$

Thus, this is a basis adapted to the splitting of $T^*M^{\mathbb{C}}$ into the eigenspaces of J^*. At some points we will use

$$\theta^j \coloneqq \mu^j + i\,\sigma^j, \quad \theta^{\bar{j}} \coloneqq \mu^j - i\,\sigma^j = \bar{\theta}^j, \tag{3.1.16}$$

which satisfy

$$J(\theta^j) = i\,\theta^j \quad \text{and} \quad J(\theta^{\bar{j}}) = -i\,\theta^{\bar{j}}. \tag{3.1.17}$$

With these definitions,

$$\Lambda^{1,0} = \operatorname{span}_{\mathbb{C}}\{\theta^j \,|\, j = 1, \dots, n\} \quad \text{and} \quad \Lambda^{0,1} = \operatorname{span}_{\mathbb{C}}\{\theta^{\bar{j}} \,|\, j = 1, \dots, n\}. \tag{3.1.18}$$

We now come to the definition of an $SU(3)$-structure in $D = 6$. As explained in section 2.1, a G-structure on M can be characterized in terms of its defining sections.

Definition 3.1.4: *An $SU(3)$-**structure on a** 6-**dimensional manifold** M is defined by a quadruple (g, J, ω, Ω), where (M, g, J) is an almost Hermitian manifold with Kähler form $\omega \in \Omega^{1,1}(M)$, and $\Omega \in \Omega^{3,0}(M)$ is a nowhere-vanishing $(3,0)$-form on M. These defining sections are subject to the algebraic relations*

$$\begin{aligned} \omega(X, Y) &= g(J(X), Y) \quad \forall\, X, Y \in \Gamma(TM) \\ \omega \wedge \Omega &= 0, \\ \Omega \wedge \bar{\Omega} &= -\frac{4i}{3}\, \omega \wedge \omega \wedge \omega. \end{aligned} \tag{3.1.19}$$

More precisely, the $SU(3)$-structure is completely determined by either of the triples (g, J, Ω), (J, ω, Ω), or (g, ω, Ω). This is because (3.1.14) determines one element of (g, J, ω) from the remaining two. The standard form of ω is given in (3.1.14) and that of Ω reads

$$\begin{aligned} \Omega &= (\beta^1 + i\,\beta^2) \wedge (\beta^3 + i\,\beta^4) \wedge (\beta^5 + i\,\beta^6) \\ &= (\mu^1 + i\,\sigma^1) \wedge (\mu^2 + i\,\sigma^2) \wedge (\mu^3 + i\,\sigma^3). \end{aligned} \tag{3.1.20}$$

We will also make use of the real and imaginary parts of

$$\Omega = \Omega^+ + i\,\Omega^-, \quad \Omega^\pm \in \Omega^{3,0}(M) \oplus \Omega^{0,3}(M). \tag{3.1.21}$$

separately. The standard components of Ω^+ and Ω^-, thus, read

$$\begin{aligned} \Omega^+ &= \beta^{135} - \beta^{146} - \beta^{236} - \beta^{245}, \\ \Omega^- &= \beta^{136} + \beta^{145} + \beta^{235} - \beta^{246}, \end{aligned} \tag{3.1.22}$$

where we adopted the shorthand notation $\beta^{ab} = \beta^a \wedge \beta^b$.

It turns out that there are several different types of $SU(3)$-structures with different geometric properties. As we indicated in section 2.2, G-structures can be classified by their intrinsic torsion, and this governs most of their geometric features. In that section we introduced the notion of intrinsic torsion of a G-structure by means of a reference connection of the ambient principal bundle. We then pointed out that this measures how the reference connection fails to preserve the defining sections of the G-structure. From this we might expect the torsion classes to be encoded in certain derivatives of these sections. In fact, in the case of $SU(3)$-structures on 6-manifolds, the torsion classes can be read off from [14, 20]

$$\mathrm{d}\omega = \frac{3}{2}\,\mathrm{Im}\Big(\big(W_1^+ - i\,W_1^-\big)\,\Omega\Big) + W_3 + W_4 \wedge \omega, \tag{3.1.23}$$

$$\mathrm{d}\Omega = \big(W_1^+ + i\,W_1^-\big)\,\omega \wedge \omega + \big(W_2^+ + i\,W_2^-\big) \wedge \omega + \Omega \wedge W_5, \tag{3.1.24}$$

where we use a form of W_5 slightly differing from that in e. g. [14]. $W_1^\pm$ are real functions, W_4 and W_5 are real 1-forms, $W_2^\pm$ are the real and imaginary part of a

(1, 1)-form, respectively, and W_3 is the real part of a (2, 1)-form. In addition, both W_2 and W_3 are primitive forms, meaning that [14]

$$\omega \lrcorner W_2 = 0 \quad \text{and} \quad \omega \lrcorner W_3 = 0. \tag{3.1.25}$$

The $\lrcorner$ is the interior multiplication, or contraction, of two forms. It is defined with respect to a Riemannian metric, and we use the conventions of [22]. That is, it

$$\mu \lrcorner \sigma \coloneqq *(\mu \wedge *\sigma), \tag{3.1.26}$$

such that for example $\beta^{12} \lrcorner \beta^{1234} = \beta^{34}$. Note that conformal rescalings leave the combination $3\,W_4 + 2\,W_5$ invariant, and that the structure is complex if and only if $W_1^{\pm} = W_2^{\pm} = 0$.
With this we can define several special types of $SU(3)$-structures.

Definition 3.1.5: *Let (g, J, ω, Ω) be an $SU(3)$-structure on a 6-dimensional manifold M.*

*(1) (g, J, ω, Ω) is **integrable**, or **Calabi-Yau**, if*

$$\mathrm{d}\omega = 0, \quad \mathrm{d}\Omega = 0 \quad \Leftrightarrow \quad W_1 = W_2 = W_3 = W_4 = W_5 = 0. \tag{3.1.27}$$

*(2) One speaks of a **half-flat** [24] $SU(3)$-structure (g, J, ω, Ω) if*

$$\mathrm{d}\omega \wedge \omega = 0, \quad \mathrm{d}\Omega^+ = 0 \quad \Leftrightarrow \quad W_1^+ = W_2^+ = W_4 = W_5 = 0. \tag{3.1.28}$$

*(3) (g, J, ω, Ω) is called **nearly Kähler** if [25, 26]*

$$\mathrm{d}\omega = -3\lambda\,\Omega^+, \quad \mathrm{d}\Omega^- = 2\lambda\,\omega \wedge \omega \tag{3.1.29}$$

for some $\lambda \in \mathbb{R}$, which is equivalent to

$$W_1^- = 2\lambda \quad \text{and} \quad W_1^+ = W_2 = W_3 = W_4 = W_5 = 0. \tag{3.1.30}$$

We add an additional note on a special type of $U(3)$-structures. On any almost Hermitian manifold (M, g, J) there exists a unique connection preserving this structure and having totally antisymmetric torsion. This connection is called the Bismut connection [27, 28].

Definition 3.1.6: ***Kähler-torsion*** *6-manifolds are characterized by the torsion of the Bismut connection being given by*

$$T = Jd\omega, \tag{3.1.31}$$

and this being the real part of a (2, 1)-form.

In this situation, the Bismut connection is then sometimes called the canonical connection of the Kähler-torsion structure, or the Kähler-torsion connection. From [28] one can see that Kähler-torsion manifolds are complex, i. e. they are a subclass of

$$W_1^{\pm} = W_2^{\pm} = 0. \tag{3.1.32}$$

In general, their structure group is U(3) rather than SU(3), in contrast to the preceding types of manifolds. Nevertheless, as an $SU(3)$-structure is defined by the data (g, J, Ω), its principal bundle is contained in an ambient $U(3)$-structure defined by (g, J). We may then ask, whether this $U(3)$-structure is Kähler-torsion, and if so, call the $SU(3)$-structure *Kähler-torsion* as well.
We have thus seen, that every $SU(3)$-structure is contained in a $U(3)$-structure. The converse, however, need not be true, as for instance if M is non-orientable. Yet, a $U(3)$-structure might be reducible to an $SU(3)$-structure, and it certainly will be if it carries a connection with holonomy $SU(3)$. In this case, we could reduce the $U(3)$-principal bundle to the holonomy bundle of that connection.
In particular, one may ask whether the Bismut connection has holonomy group $SU(3)$. For a given $SU(3)$-structure, we may even consider its ambient $U(3)$-structure and investigate whether the Bismut connection restricts to a connection on the $SU(3)$ principal bundle. This is non-trivial as elaborated on in section 2.2 and turns out to hold true if the $SU(3)$-structure is Kähler-torsion in the aforementioned sense and additionally satisfies

$$2\,W_4 + W_5 = 0. \tag{3.1.33}$$

This result has been obtained for example in [29]. $SU(3)$-structures which are Kähler-torsion in the above sense and additionally satisfy $2\,W_4 + W_5 = 0$ are called *Calabi-Yau-torsion*. These are the Kähler-torsion $SU(3)$-structures for which the Bismut connection of the ambient $U(3)$-structure is compatible with the $SU(3)$-structure.
In chapter 5, we construct and examine Kähler-torsion, nearly-Kähler, and half-flat $SU(3)$-structures on 6-manifolds from suited $SU(2)$-structures in $D = 5$.

3.2 SU(2)-Structures in D = 5

We introduce $SU(2)$-structures on 5-manifolds and examine special examples in this and the following section.
$SU(2)$-structures on 5-manifolds can be characterized in terms of defining sections as introduced in section 2.1, similar to the $SU(3)$-structures in 6 dimensions. For $SU(2)$-structures on 5-manifolds, this has been worked out in [30]. There the following result has been proven:

Proposition 3.2.1: *$SU(2)$-structures on 5-dimensional manifolds M^5 are in one-to-one correspondence to quadruples $(\eta, \omega^1, \omega^2, \omega^3)$, which consist of $\eta \in \Omega^1(M)$ and $\omega^i \in \Omega^2(M)$ $\forall\, i = 1, 2, 3$ satisfying*

$$\omega^i \wedge \omega^j = 2\,\delta^{ij}\,Q \;\; \text{for}\; Q = \frac{1}{2}\,\omega^3 \wedge \omega^3 \in \Omega^4(M), \tag{3.2.1}$$

$$\eta \wedge Q \neq 0 \quad \text{and} \tag{3.2.2}$$

$$\iota_X \omega^3 = -\iota_Y \omega^2 \Rightarrow \omega^1(X, Y) \geqslant 0. \tag{3.2.3}$$

Note that our choice of order and signs of the forms differ from that in [30]but the formulas are adapted accordingly. The main reason for our choice will become obvious from the next proposition.
Equivalently, the defining sections single out frames of TM that are elements of a principal $SU(2)$-subbundle of $F(TM)$. As presented in proposition 2.1.8, the $SU(2)$-structure consists of precisely those frames, in which the defining sections (η, ω^i) have certain standard components. These components have first been written down in [30], and with a slight relabeling and change of the sign of β^5 the result is

Proposition 3.2.2: *$SU(2)$-structures on 5-dimensional manifolds M^5 are in one-to-one correspondence to quadruples $(\eta, \omega^1, \omega^2, \omega^3)$, consisting of forms $\eta \in \Omega^1(M)$ and $\omega^i \in \Omega^2(M)$, $i = 1, 2, 3$ such that around every $x \in M$ there is some open neighborhood $U \subset M$ and a local frame $e \in \Gamma_U(F(TM))$ with dual coframe β, expressed in which $(\eta, \omega^1, \omega^2, \omega^3)$ take the standard form*

$$\begin{aligned}
\eta &= -\beta^5, \\
\omega^1 &= \beta^1 \wedge \beta^4 + \beta^2 \wedge \beta^3, \\
\omega^2 &= -\beta^1 \wedge \beta^3 + \beta^2 \wedge \beta^4, \\
\omega^3 &= \beta^1 \wedge \beta^2 + \beta^3 \wedge \beta^4.
\end{aligned} \tag{3.2.4}$$

With this convention for the defining forms and making use of the 't Hooft tensor

$$\eta^a{}_{bc} := \epsilon^a{}_{bc4} + \delta^a_b\,\delta_{c4} - \delta^a_c\,\delta_{b4}, \quad a, b, c = 1, \ldots, 4 \tag{3.2.5}$$

(not to be confused with the 1-form η) as in [31], we can unify the standard forms of the ω^i as

$$\omega^\alpha = \frac{1}{2}\,\eta^\alpha{}_{bc}\,\beta^b \wedge \beta^c, \quad \alpha = 1, 2, 3. \tag{3.2.6}$$

Note that since $SU(2)$ can be embedded into $SO(5)$, there also is a Riemannian metric g on M which takes its standard form in the bases that constitute the $SU(2)$-structure.

Similar to the case of $SU(3)$-structures in $D = 6$, there are special types of $SU(2)$-structures in $D = 5$. These are singled out by certain conditions on the differentials of the defining forms. We define them as in [24], but with signs adapted.

Definition 3.2.3: *Consider an $SU(2)$-structure on a 5-dimensional manifold M^5 defined by $(\eta, \omega^1, \omega^2, \omega^3)$.*

*(1) The $SU(2)$-structure is said to be **hypo** if*

$$\mathrm{d}\omega^3 = 0, \quad \mathrm{d}(\eta \wedge \omega^1) = 0, \quad \textit{and} \quad \mathrm{d}(\eta \wedge \omega^2) = 0. \tag{3.2.7}$$

*(2) It is called **nearly hypo** if*

$$\mathrm{d}\omega^1 = -3\,\eta \wedge \omega^2, \quad \textit{and} \quad \mathrm{d}(\eta \wedge \omega^3) = 2\,\omega^1 \wedge \omega^1. \tag{3.2.8}$$

*(3) The $SU(2)$-structure is **double hypo** if it is hypo and nearly hypo.*

*(4) An $SU(2)$-structure is defined to be **Sasaki-Einstein** if*

$$\mathrm{d}\eta = 2\,\omega^3, \quad \mathrm{d}\omega^1 = -3\,\eta \wedge \omega^2, \quad \textit{and} \quad \mathrm{d}\omega^2 = 3\,\eta \wedge \omega^1. \tag{3.2.9}$$

Clearly Sasaki-Einstein $SU(2)$-structures are double hypo, but the converse does not necessarily hold true.

After this exposition of $SU(3)$- and $SU(2)$-structures in $D = 6$ and $D = 5$, respectively, we go more into the details of Sasakian and Sasaki-Einstein manifolds. Study of the former structures will provide us with valuable additional knowledge about the rich structure of these manifolds. The richer a structure there is on a manifold, the more objects there are whose effects can be interwoven in explicit constructions. However, we must not be too restrictive in order not to rule out too many phenomenologically viable classes of manifolds.

3.3 Sasakian Structures

First, we collect the characterizations of Sasakian structures used in e. g. [17] or [32]. We take from [17] the following two definitions and the subsequent theorem, adapting them to our conventions for differential forms, which coincide with the ones of [32].

Definition 3.3.1: *We define the following geometric structures on a manifold M^D of dimension $D = 2n + 1$:*

*(1) First, a **contact structure** is defined by a 1-form $\eta \in \Omega^1(M)$ having the property that*

$$\big(\eta \wedge (\mathrm{d}\eta)^n\big)_{|x} \neq 0 \quad \forall\, x \in M. \tag{3.3.1}$$

*(2) An **almost contact structure** on M is given by a triple (η, ξ, Φ), where $\eta \in \Omega^1(M)$, $\xi \in \Gamma(TM)$ and $\Phi \in End(TM)$, satisfying the relations*

$$\eta(\xi) = 1 \quad \textit{and} \quad \Phi^2 = -\mathbb{I} + \eta \otimes \xi. \tag{3.3.2}$$

*(3) We define a **metric almost contact structure** on M to be a tuple (g, η, ξ, Φ), where (η, ξ, Φ) is an almost contact structure on M and (M, g) is a Riemannian manifold compatible with the almost contact structure in the sense that*

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\,\eta(Y) \quad \forall\, X, Y \in \Gamma(TM). \tag{3.3.3}$$

*(4) A metric almost contact structure is called **normal** if the lift of Φ to the metric cone over M induces an integrable almost complex structure. This turns out to be equivalent to*

$$\mathcal{L}_\xi\, \Phi = 0. \tag{3.3.4}$$

*(5) A **metric contact structure** on M is a metric almost contact structure (g, η, ξ, Φ) on M such that η is a contact form with*

$$\mathrm{d}\eta = 2\, g\big(\Phi(\cdot), \cdot\big). \tag{3.3.5}$$

*(6) A metric contact structure is defined to be **K-contact** if its Reeb vector field is Killing.*

*(7) Finally, a normal metric contact structure is called a **Sasakian structure**.*

Note that we use the conventions of [32] regarding the exterior differential and wedge product, thus requiring the 2 in (3.3.5).
In particular, we see from this definition that on a metric almost contact manifold ξ always is of unit length. Thus, around every point $x \in M$ there is a local orthonormal frame such that

$$\xi = -e_D, \quad \eta = -\beta^D, \quad \text{and} \quad \Phi = \sum_{j=1}^{n} \beta^{2j-1} \otimes e_{2j} - \beta^{2j} \otimes e_{2j-1}. \tag{3.3.6}$$

Also, $\xi \perp_g ker(\eta)$.
Other characterizations of Sasakian manifolds that may be helpful are the following [17].

Proposition 3.3.2: *Let (M, g) be a Riemannian manifold and ∇^g its Levi-Civita connection. Then the following properties are equivalent:*

(1) There exists a Killing vector field ξ of unit length on M such that the tensor field $\Phi \in End(TM)$, $\Phi(X) \coloneqq \nabla^g_X \xi$ satisfies

$$(\nabla^g_X \Phi)(Y) = g(\xi, Y)\, X - g(X, Y)\, \xi \quad \forall\, X, Y \in \Gamma(TM). \tag{3.3.7}$$

(2) *There exists a Killing vector field* ξ *of unit length on* M *such that the curvature* 2*-form* $R \in \Omega^2(M, End(TM))$ *of* ∇^g *satisfies*

$$R(X, \xi)(Y) = g(\xi, Y)\, X - g(X, Y)\, \xi \quad \forall\, X, Y \in \Gamma(TM). \tag{3.3.8}$$

(3) *The metric cone* $(C(M), \bar{g}) = (M \times \mathbb{R}_+, \mathrm{d}r^2 + r^2\, g)$ *with* $\omega := \frac{r^2}{2}\, \mathrm{d}\eta + r\, \mathrm{d}r \wedge \eta$ *is Kähler.*

(4) (M, g) *is Sasakian.*

Depending on the situation, we can choose one of the above equivalent characterizations in order to check whether or not a manifold is Sasakian. But we can also prove another characterization of Sasakian structures, which is similar to the known above ones, but might be interesting nevertheless. This will be of help when searching for the connections with reduced holonomy on Sasakian manifolds. To make the proof of this characterization easier to read, we state two technical lemmas first:

Lemma 3.3.3: *Be* (M, g, ξ, η, Φ) *a manifold endowed with a metric contact structure, and be* $(C_\phi(M), \bar{g}) := (M \times I, \mathrm{d}r^2 + \phi(r)^2\, g)$ *its* ϕ*-cone.* $I \subset \mathbb{R}$ *is some (open) interval, on which the smooth function* $\phi : I \to \mathbb{R}_+$, $r \mapsto \phi(r)$ *is defined.*
In the fashion of [32], denote by $\Psi_{|(r,x)} = \phi(r)\, \partial_{r|(r,x)}$ *the adaption of the Euler vector field to* $C_\phi(M)$ *and by* Ψ' *its dual* 1*-form. Define*

$$J = \Phi \oplus (\Psi' \otimes \xi - \eta \otimes \Psi) = \Phi \oplus \left(\mathrm{d}r \otimes \frac{1}{\phi}\, \xi - \phi\, \eta \otimes \mathrm{d}r\right). \tag{3.3.9}$$

Then, $(C_\phi(M), \bar{g}, J)$ *is an almost Hermitian manifold and its Kähler form is given by [32]*

$$\omega = \frac{1}{2}\, \phi^2\, \mathrm{d}\eta + \phi\, \mathrm{d}r \wedge \eta. \tag{3.3.10}$$

Proof. It is immediate that $J^2 = -\mathbb{I}$. Be $X, Y \in \Gamma(ker(\eta))$. We compute

$$\begin{aligned}
\bar{g}(J(X), J(Y)) &= \phi^2\, g(\Phi(X), \Phi(Y)) = \phi^2\, g(X, Y) = \bar{g}(X, Y),\\
\bar{g}(J(\xi), J(X)) &= 0 = \bar{g}(\xi, X),\\
\bar{g}(J(\Psi), J(X)) &= \phi^2\, g(\xi, \Phi(X)) = 0 = \bar{g}(\Psi, X),\\
\bar{g}(J(\xi), J(\Psi)) &= 0 = \bar{g}(\xi, \Psi),\\
\bar{g}(J(\xi), J(\xi)) &= \phi^2 = \bar{g}(\xi, \xi) \quad \text{and}\\
\bar{g}(J(\Psi), J(\Psi)) &= \phi^2 = \bar{g}(\Psi, \Psi).
\end{aligned} \tag{3.3.11}$$

Thus, $(C(M), g, J)$ is an almost Hermitian manifold, and its Kähler form is generically given by

$$\omega(\cdot, \cdot) = g(J(\cdot), \cdot). \tag{3.3.12}$$

Be $X, Y \in \Gamma(ker(\eta))$. On the ϕ-cone of M this yields

$$\begin{aligned}
\omega(X,Y) &= \bar{g}(J(X),Y) = \bar{g}(\Phi(X),Y) = \phi^2\, g(\Phi(X),Y) \\
&= \frac{\phi^2}{2}\,\mathrm{d}\eta\,(X,Y) = \left(\phi\,\mathrm{d}r \wedge \eta + \frac{\phi^2}{2}\,\mathrm{d}\eta\right)(X,Y), \\
\omega(\xi,X) &= \bar{g}(J(\xi),X) = -\bar{g}(\Psi,X) = 0 \\
&= \left(\phi\,\mathrm{d}r \wedge \eta + \frac{\phi^2}{2}\,\mathrm{d}\eta\right)(\xi,X), \\
\omega(\Psi,X) &= \bar{g}(\xi,X) = 0 = \left(\phi\,\mathrm{d}r \wedge \eta + \frac{\phi^2}{2}\,\mathrm{d}\eta\right)(\Psi,X),
\end{aligned} \tag{3.3.13}$$

and, finally,

$$\omega(\xi,\Psi) = \bar{g}(J(\xi),\Psi) = -\phi(r)^2 = \left(\phi\,\mathrm{d}r \wedge \eta + \frac{\phi^2}{2}\,\mathrm{d}\eta\right)(\xi,\Psi). \tag{3.3.14}$$

Hence, for this choice of a complex structure on $C_\phi(M)$,

$$\omega = \phi\,\mathrm{d}r \wedge \eta + \frac{\phi^2}{2}\,\mathrm{d}\eta. \tag{3.3.15}$$

This proves the statement. □

From the expression for ω we see that it will be a closed 2-form for $\phi(r) = r$. However, J is not integrable in general, whence the metric cone fails to be Kähler in the generic case. This illustrates, how Sasakian manifolds are more special than metric contact manifolds.
The second lemma is generalized from [32] to the ϕ-cone, as it is stated there for the metric cone only, and we aim to consider sine-cones, for instance, later on.

Lemma 3.3.4: *Let (M,g) be a Riemannian manifold and $(C_\phi(M),\bar{g})$ its ϕ-cone. Then, for $X, Y \in \Gamma(TM)$ and Ψ as above, we have*

$$\begin{aligned}
&\nabla^{\bar{g}}_{\Psi}\,\Psi = \partial_r\phi\,\Psi, \\
&\nabla^{\bar{g}}_{X}\,\Psi = \nabla^{\bar{g}}_{\Psi}\,X = \partial_r\phi\,X \;\; \textit{and} \\
&\nabla^{\bar{g}}_{X}\,Y = \nabla^{g}_{X}\,Y - \partial_r\phi\,g(X,Y)\,\Psi.
\end{aligned} \tag{3.3.16}$$

Proof. With $[\Psi, X] = 0$ and $\mathrm{d}\phi(X) = X(\phi) = 0\ \forall\, X \in \Gamma(TM)$, we obtain from Koszul's formula that

$$\begin{aligned}
2\,\bar{g}(\nabla^{\bar{g}}_{X}\,Y,Z) &= 2\,\phi^2\,g(\nabla^{g}_{X}\,Y,Z) = 2\,\bar{g}(\nabla^{g}_{X}\,Y,Z), \\
2\,\bar{g}(\nabla^{\bar{g}}_{X}\,Y,\partial_r) &= -\,\mathcal{L}_{\partial_r}(\phi^2\,g(X,Y)) = -2\,\phi\,\partial_r\phi\,g(X,Y).
\end{aligned} \tag{3.3.17}$$

From this we take

$$\nabla^{\bar{g}}_X Y = \nabla^g_X Y - \partial_r\phi\, g(X,Y)\,\Psi. \tag{3.3.18}$$

Furthermore, also from Koszul's fomrula,

$$\begin{aligned} 2\,\bar{g}(\nabla^{\bar{g}}_\Psi \Psi, \Psi) &= \mathcal{L}_\Psi(\phi^2) = 2\,\phi^2\,\partial_r\phi = 2\,\partial_r\phi\, g(\Psi,\Psi), \\ 2\,\bar{g}(\nabla^{\bar{g}}_\Psi \Psi, X) &= 0, \\ 2\,\bar{g}(\nabla^{\bar{g}}_\Psi X, \Psi) &= 0, \\ 2\,\bar{g}(\nabla^{\bar{g}}_\Psi X, Y) &= \mathcal{L}_\Psi(\phi^2\, g(X,Y)) = 2\,\partial_r\phi\,\bar{g}(X,Y). \end{aligned} \tag{3.3.19}$$

Hence,

$$\begin{aligned} &\nabla^{\bar{g}}_\Psi X = \partial_r\phi\, X = \nabla^{\bar{g}}_X \Psi \quad \text{and} \\ &\nabla^{\bar{g}}_\Psi \Psi = \partial_r\phi\,\Psi. \end{aligned} \tag{3.3.20}$$

This proves the lemma. □

We now come to the alternative characterization of Sasakian manifolds.

Proposition 3.3.5: *A metric contact structure is Sasakian if and only if it is K-contact and satisfies*

$$\nabla^g_X(\mathrm{d}\eta) = 2\,\flat_g(X)\wedge\eta \quad \forall\, X\in\Gamma(TM). \tag{3.3.21}$$

Proof. Assume we are given a metric contact structure (g,ξ,η,Φ) on M. Recall that these defining sections satisfy

$$\Phi(\xi) = 0 \quad \text{and} \quad \mathrm{d}\eta\,(X,Y) = 2\,g(\Phi(X),Y) \quad \forall\, X,Y\in\Gamma(TM). \tag{3.3.22}$$

Let us consider the metric cone $(C(M),\bar{g}) = (M\times\mathbb{R}_+, \mathrm{d}r^2 + r^2\,g)$ over (M,g). It is endowed with the Euler vector field Ψ, almost complex structure J, and Kähler form ω as constructed in lemma 3.3.3. From this it is clear that $(C\,(M),\bar{g},J)$ is an almost Hermitian manifold. Moreover, recall that ω constructed this way is closed on the metric cone. Thus, we are left to prove the integrability of J.
We may, however, take another route and use the fact that the structure on M is Sasakian if and only if its cone is Kähler (see proposition 3.3.2). For example from [17], proposition 3.1.9, we take that an almost Hermitian manifold (M,J,g) is Kähler if and only if its Kähler 2-form is parallel with respect to the Levi-Civita connection of g. We, therefore, investigate under which conditions $\nabla^{\bar{g}}\omega = 0$.
The covariant derivative $\nabla^{\bar{g}}\omega$ is defined by means of the Leibniz rule on the tensor algebra, i. e. for all $X,Y,Z\in\Gamma(T\,C(M))$ we have

$$\mathcal{L}_X(\omega(Y,Z)) = (\nabla^{\bar{g}}_X\omega)(Y,Z) + \omega(\nabla^{\bar{g}}_X Y, Z) + \omega(Y,\nabla^{\bar{g}}_X Z). \tag{3.3.23}$$

Consider $X, Y, Z \in \Gamma(TM)$. Using (3.3.10) and lemma 3.3.5 we compute

$$\begin{aligned}(\nabla^{\bar g}_{\partial_r}\omega)(Y,Z) &= \mathcal{L}_{\partial_r}\left(\frac{r^2}{2}\,\mathrm{d}\eta\,(Y,Z)\right) - \frac{1}{r}\,\omega(\nabla^{\bar g}_{\Psi}Y,Z) - \frac{1}{r}\,\omega(Y,\nabla^{\bar g}_{\Psi}Z)\\ &= r\,\mathrm{d}\eta\,(Y,Z) - \frac{1}{2r}\,\mathrm{d}(r^2\,\eta)\,(Y,Z) - \frac{1}{2r}\,\mathrm{d}(r^2\,\eta)\,(Y,Z)\\ &= 0,\end{aligned} \tag{3.3.24}$$

$$\begin{aligned}(\nabla^{\bar g}_{\partial_r}\omega)(Y,\Psi) &= \mathcal{L}_{\partial_r}\big(r\,\mathrm{d}r\wedge\eta\,(Y,\Psi)\big) - \frac{2}{r}\,\omega(Y,\Psi)\\ &= -2\,r\,\eta(Y) - \frac{2}{r}\,(r\,\mathrm{d}r\wedge\eta\,(Y,\Psi))\\ &= 0,\end{aligned} \tag{3.3.25}$$

$$\begin{aligned}(\nabla^{\bar g}_{X}\omega)(Y,Z) &= \mathcal{L}_{X}\left(\frac{r^2}{2}\,\mathrm{d}\eta\,(Y,Z)\right)\\ &\quad - \omega(\nabla^{g}_{X}Y - g(X,Y)\Psi, Z) - \omega(Y,\nabla^{g}_{X}Z - g(X,Z)\Psi)\\ &= \frac{r^2}{2}\,(\nabla^{g}_{X}\,\mathrm{d}\eta)(Y,Z) - g(X,Y)\,\omega(\Psi,Z) + g(X,Z)\,\omega(Y,\Psi)\\ &= \frac{r^2}{2}\,(\nabla^{g}_{X}\,\mathrm{d}\eta)(Y,Z) - r^2\,g(X,Y)\,\eta(Z) + r^2\,g(X,Z)\,\eta(Y)\\ &= \frac{r^2}{2}\,\Big((\nabla^{g}_{X}\,\mathrm{d}\eta) + 2\,\eta\wedge\flat_g(X)\Big)(Y,Z),\end{aligned} \tag{3.3.26}$$

$$\begin{aligned}(\nabla^{\bar g}_{X}\omega)(\Psi,Z) &= \mathcal{L}_{X}(r\,\mathrm{d}r\wedge\eta\,(\Psi,Z))\\ &\quad - \omega(X,Z) - \omega(\Psi,\nabla^{g}_{X}Z - g(X,Z)\,\Psi)\\ &= r^2\mathcal{L}_{X}\,(\eta(Z)) - \frac{r^2}{2}\,\mathrm{d}\eta\,(X,Z) - r\,\mathrm{d}r\wedge\eta\,(\Psi,\nabla^{g}_{X}Z)\\ &= r^2\mathcal{L}_{X}\,(\eta(Z)) - \frac{r^2}{2}\,\mathrm{d}\eta\,(X,Z) - r^2\,\eta(\nabla^{g}_{X}Z).\end{aligned} \tag{3.3.27}$$

Here we make use of $\eta(X) = g(\xi, X)\ \forall\, X \in \Gamma(TM)$ and

$$\mathrm{d}\mu\,(X,Z) = X(\mu(Z)) - Z(\mu(X)) - \mu([X,Z])\ \forall\, X,Z \in \Gamma(TM),\ \mu \in \Omega^1(M), \tag{3.3.28}$$

as well as Koszul's formula. Thereby, we obtain

$$(\nabla^{\bar g}_{X}\omega)(\Psi,Z) = r^2\Big(\mathcal{L}_X(g(\xi,Z)) - \frac{1}{2}\Big(X(\eta(Z)) - Z(\eta(X)) - \eta([X,Z]) \tag{3.3.29}$$

$$+ \mathcal{L}_X\,(g(\xi,Z)) + \mathcal{L}_Z\,(g(\xi,X)) - \mathcal{L}_\xi\,(g(X,Z)) \tag{3.3.30}$$

$$- g(X,[Z,\xi]) + g(Z,[\xi,X]) + g(\xi,[X,Z])\Big)\Big) \tag{3.3.31}$$

$$= \frac{r^2}{2}\Big(\mathcal{L}_\xi(g(X,Z)) - g(X,\mathcal{L}_\xi Z) - g(\mathcal{L}_\xi X, Z)\Big) \tag{3.3.32}$$

$$= \frac{r^2}{2}\,(\mathcal{L}_\xi\, g)(X,Z). \tag{3.3.33}$$

Thus, the almost Hermitian structure on the cone is Kähler *if and only if* ξ is Killing and

$$(\nabla^g_X \,\mathrm{d}\eta) = 2\,\flat_g(X) \wedge \eta \quad \forall\, X \in \Gamma(TM). \tag{3.3.34}$$

Therefore, the properties demanded in the statement of the proposition are equivalent to M being Sasakian. □

The reason that this characterization might be interesting is that from the properties stated in the proposition one can very directly construct a connection preserving the Sasakian structure. This yields the characteristic connection of Sasakian manifolds introduced in [28]. As the Levi-Civita connection of g does not preserve the structure, one can, in particular, infer that Sasakian structures are not integrable in the sense of section 2.2. This is reflected in the fact that Sasaki-Einstein structures can be defined by Killing spinors as defining sections. As their Killing constants are $\pm\frac{1}{2}$ [17], these are not parallel with respect to ∇^g.
Additionally, note that generically Sasakian structures on $(D = 2n+1)$-dimensional spaces are $U(n)$-structures rather than $SU(n)$-structures [17]. We will make use of the latter in our constructions later on.

3.4 Example: The 5-Sphere

The 5-sphere S^5 provides an example of a manifold which is non-trivial on the one hand, but which carries much geometric structure on the other. Therefore, it is a good candidate for model building of mathematical structures as well as for special solutions to higher-dimensional physics.
The 5-sphere can be represented as the coset space [33]

$$S^5 = \frac{SU(3)}{SU(2)}. \tag{3.4.1}$$

This can be seen as follows: $SU(3)$ acts smoothly and transitively on the set of 3-dimensional complex vectors of unit norm [33], which can be identified with S^5. The stabilizer of, for example, the vector $v = (0,0,1) \in \mathbb{C}^3$ under this $SU(3)$-action are those elements of $SU(3)$ having $(0,0,1)$ in their first column. For such a matrix to be unitary, it is necessary that these also are the entries of its first row. So, it must be a block-diagonal matrix $(A,1) \in SU(3)$, which requires $A \in SU(2)$. Since every matrix of this kind leaves v invariant, $SU(2)$ is the stabilizer of the $SU(3)$-action under consideration and S^5 does indeed coincide with the homogeneous space

$SU(3)/SU(2)$.
Because of this way of describing S^5, we may employ some of the mathematical structure of cosets, which we now turn our attention to. First, it is important that every coset $M = H/G$ can be seen as the base of the principal bundle $(H, \pi, H/G, G)$. On the total space, namely on the Lie group H, elements of $\mathfrak{h}$ give rise to globally defined, left-invariant vector fields and their dual left-invariant 1-forms. We can pull these 1-forms back to the base in order to obtain local coframes with special properties. For this purpose, be $\{(U_\lambda, \sigma_\lambda)\}_{\lambda\in\Lambda}$ a covering of M with local sections of $(H, \pi, H/G, G)$. Let $I_1, \ldots, I_{dim(H)}$ be a basis of $\mathfrak{h}$ and $\theta^1, \ldots, \theta^{dim(H)}$ the induced left-invariant 1-forms on H.
The Lie algebra of H splits into $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ as in section 2.2. We arrange the generators of $\mathfrak{h}$ such that $\mathrm{m} = span_{\mathbb{R}}\{I_1, \ldots, I_D\}$ and let small indices $a, b, \ldots$ run from 1 to D, as well as small indices $i, j, \ldots$ from $D+1$ to $dim(H)$. Capital indices $A, B, \ldots$ are assumed to run from 1 to $dim(H)$. Then the pullback of the coframe on H to M is given by $\beta_\lambda^A \coloneqq {\sigma_\lambda}^*\, \theta^A$.
For the β_λ^a to yield local coframes on M, we should first establish their linear independence at every point of U_λ. This is achieved by recalling that the σ_λ are local sections of the principal bundle H, thus implying $\pi^*\, \beta_\lambda^a = (\sigma_\lambda \circ \pi)^*\, \theta^a = \theta^a$ on G. Therefore, $dim(span_{\mathbb{R}}\{\beta_\lambda^1, \ldots, \beta_\lambda^D\}) = D$.
In order to construct global fields on M from the local pullbacks of these forms, we need the relation between the $\beta_\lambda^A = {\sigma_\lambda}^*\, \theta^A$ with respect to different local sections σ_λ of H. Let us, therefore, briefly investigate the transformation behavior of the β_λ^A upon change of σ_λ. The general result is

Lemma 3.4.1: *Consider a homogeneous space $M^D = H/G$ and $\beta_\lambda^A \coloneqq {\sigma_\lambda}^*\, \theta^A$ as described above. Then, for $\rho, \lambda \in \Lambda$ such that $U_{\rho\lambda} \neq \varnothing$ and $\sigma_\rho = R_g \circ \sigma_\lambda$ on $U_{\rho\lambda}$, we have*

$$\beta_{\rho\,|x}^A = Ad_H\big(g(x)^{-1}\big)^A{}_B\; \beta_{\lambda\;|x}^B + \theta^A \circ \varphi \circ g^* \mu_{G\,|x}, \tag{3.4.2}$$

where $\varphi : \mathfrak{g} \to \Gamma(T^v H)$ assigns to $\xi \in \mathfrak{g}$ the fundamental vector field on the principal bundle, and μ_G is the Maurer-Cartan 1-form on G.

Proof. From [18] we take the formula

$$(R_g \circ \sigma_\rho)_{*\,|x} = (R_{g(x)})_* \circ \sigma_{\rho\,*|x} + \varphi \circ g^* \mu_{G\,|x}. \tag{3.4.3}$$

Thus,

$$\begin{aligned} \beta_{\rho\,|x}^A &= ({\sigma_\rho}^*\, \theta^A)_{|x} = \theta^A \circ (R_g \circ \sigma_\lambda)_{*\,|x} \\ &= \big((R_{g(x)})^*\, \theta^A\big) \circ \sigma_{\lambda *\,|x} + \theta^A \circ \varphi \circ g^* \mu_{G\,|x}. \end{aligned} \tag{3.4.4}$$

Denote by $E_1, \ldots, E_{dim(H)}$ the set of left-invariant vector fields on H generated by the I_A. Since the right-action of G on the principal bundle coincides with the right multiplication on H, we may compute

$$\begin{aligned} R_{g*}\,(E_{A|h}) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|0} R_{\exp(t\,I_A)g}\,h = \frac{\mathrm{d}}{\mathrm{d}t}_{|0} R_{g\,g^{-1}\exp(t\,I_A)g}\,h \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|0} R_{\exp(Ad_H(g^{-1})(t\,I_A))}\,(hg) = \frac{\mathrm{d}}{\mathrm{d}t}_{|0} R_{\exp(t\,Ad_H(g^{-1})^B{}_A\,I_B)}\,(hg) \\ &= Ad_H(g)^B{}_A\,E_{B|hg}. \end{aligned} \tag{3.4.5}$$

This is the left-invariant vector field associated to $Ad_G(g)(I_A)$.
Now consider the dual 1-form. Because of

$$(R_g^*\,\theta^A)_{|h}(E_B) = \theta^A(R_{g*}\,(E_{B|h})) = Ad_H\big(g(x)^{-1}\big)^A{}_B \tag{3.4.6}$$

we must have

$$R_g^*\,\theta^A = Ad_H\big(g(x)^{-1}\big)^A{}_B\,\theta^B. \tag{3.4.7}$$

By the linearity of the pullback, the statement follows. □

Corollary 3.4.2: *In the situation of the preceding lemma, be I_a, $a = 1, \dots, D$ a basis of* m*, where $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$. The left-invariant* 1*-forms associated to these generators transform as*

$$\beta^a_{\rho\,|x} = Ad_H\big(g(x)^{-1}\big)^a{}_B\;\beta^B_{\lambda\;|x}, \tag{3.4.8}$$

where also $B = 1, \dots, dim(H)$.

Proof. This is because the left-invariant vector field generated by $\xi \in \mathfrak{g} \subset \mathfrak{h}$ on the Lie group H coincides with the fundamental vector field $\varphi(\xi)$, generated by ξ on the principal fiber bundle H. The reason for that is that already the right multiplications by elements of G coincide on H for the two points of view.
Furthermore, every θ^a, $a = 1, \dots, D$ vanishes on E_A as long as $A \neq a$. Recall that the structure group of the principal bundle under consideration is G, whence $g^*\mu_G$ takes values in $\mathfrak{g}$ exclusively. Therefore, its composition with φ yields vector fields that are linear combinations of the E_i only, whence

$$\theta^a \circ \varphi \circ g^*\mu_{G\,|x} = 0. \tag{3.4.9}$$

This leaves us with the purely homogeneous transformation law. □

For a reductive homogeneous space and connected G this can be restricted further to

$$\beta^a_{\rho\,|x} = Ad_H\big(g(x)^{-1}\big)^a{}_b\;\beta^b_{\lambda\;|x}. \tag{3.4.10}$$

Thus, we have established that the β^a_λ may serve as local coframes on M, and we also have examined the relation between different coframes of this type. In the reductive case we have seen that there exist local covers of $M = H/G$ with local coframes β_ρ that are related by transformations matrices from $Ad_H(G)$. Hence, we have

Corollary 3.4.3: *A reductive homogeneous space $M = H/G$ is naturally endowed with a G-structure. The group homomorphism from G to $GL(D,\mathbb{R})$ is given by $g \mapsto (Ad_H(g))_{|\mathrm{m}}$.*

Now we would like to go one step further. That is, for compact Lie algebra $\mathfrak{h}$ the local coframes constructed above can be used to define a pseudo-Riemannian metric on M.
To see this, note that a compact Lie algebra splits as $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$, where now $\mathrm{m} = \mathfrak{g}^{\perp_K}$ with respect to the non-degenerate Cartan-Killing form K on $\mathfrak{h}$. We may then choose $(I_1, \ldots, I_{dim(H)})$ to be an orthonormal basis for K. Since K is preserved by the adjoint representation, and since for reductive splittings $\mathfrak{h} = \mathfrak{g} \oplus \mathrm{m}$ both of the subspaces of $\mathfrak{h}$ are invariant under Ad_H restricted to G, the matrices $Ad_H(g)^a{}_b$ lie in $O(p,q)$, where (p,q) is the signature of K restricted to $\mathfrak{g}$.
Hence, the transformations (3.4.10), linking different coframes of the above kind to each other, are in fact orthonormal transformations. This implies that the set $\{(\beta^1_\lambda, \ldots, \beta^D_\lambda)\}_{\lambda\in\Lambda}$ defines a principal $O(p,q)$-subbundle of $F(T^*M)$ and, therefore, a pseudo-Riemannian metric of signature (p,q) on M.

After these general considerations we turn our attention back to the example of $S^5 = SU(3)/SU(2)$. For the above considerations to apply directly, we should start by isolating $\mathfrak{su}(2)$ as contained in $\mathfrak{su}(3)$.
To this end, we can choose the $\mathfrak{su}(3)$ generators $\tilde{I}_1, \ldots, \tilde{I}_8$ such that the structure constants become

$$\begin{aligned} &\tilde{f}_{31}{}^6 = -\tilde{f}_{24}{}^6 = \tilde{f}_{14}{}^7 = -\tilde{f}_{23}{}^7 = \tilde{f}_{12}{}^8 = -\tilde{f}_{34}{}^8 = \frac{1}{2\sqrt{3}}, \\ &\tilde{f}_{67}{}^8 = \frac{1}{\sqrt{3}}, \\ &\tilde{f}_{12}{}^5 = \tilde{f}_{34}{}^5 = -\frac{1}{2}. \end{aligned} \tag{3.4.11}$$

These are completely antisymmetric with respect to permutations of all the indices, since this is an orthonormal basis for the Cartan-Killing form of $\mathfrak{su}(3)$ as used in the general setup before. That is, $\tilde{f}_{AB}{}^C = -\tilde{f}_{AC}{}^B$ and so on, where $A, B, C = 1, \ldots, 8$. All structure constants with indices that are not a permutation of the ones listed above vanish. The $\mathfrak{su}(2)$-subalgebra is spanned by $\tilde{I}_6, \tilde{I}_7$ and $\tilde{I}_8$.
Now, we introduce a two-parameter family of $SU(2)$-structures on S^5 by rescaling the generators of $\mathfrak{su}(3)$. Consider

$$I_a \to \tilde{I}_a = \frac{1}{\delta} I_a\,, \qquad I_5 \to \tilde{I}_5 = \frac{1}{\gamma} I_5, \qquad I_i \to \tilde{I}_i = I_i \tag{3.4.12}$$

for $(\gamma, \delta) \in \mathbb{R}_+ \times (\mathbb{R}\backslash\{0\})$. (Flipping the sign of γ does not define a different $SU(2)$-

structure.) The structure constants are changed as follows:

$$\begin{aligned}
&\tilde{f}_{5a}{}^{b} \mapsto f_{5a}{}^{b} = \frac{1}{\gamma}\,\tilde{f}_{5a}{}^{b}, && \tilde{f}_{ab}{}^{5} \to f_{ab}{}^{5} = \frac{\gamma}{\delta^2}\,\tilde{f}_{ab}{}^{5},\\
&\tilde{f}_{ab}{}^{i} \mapsto f_{ab}{}^{i} = \frac{1}{\delta^2}\,\tilde{f}_{ab}{}^{i}, && \tilde{f}_{ia}{}^{b} \mapsto f_{ia}{}^{b} = \tilde{f}_{ia}{}^{b},\\
&\tilde{f}_{ij}{}^{k} \mapsto f_{ij}{}^{k} = \tilde{f}_{ij}{}^{k}.
\end{aligned} \tag{3.4.13}$$

A rescaling of the generators of $\mathfrak{su}(3)$ rescales the left-invariant vector fields and 1-forms on $SU(3)$ accordingly, and this is propagated to the coset via the pullback as used before.

From the structure constants (3.4.11), we see that this example is indeed reductive. Moreover, $SU(2)$ is connected (it is well known that $SU(2) \simeq S^3$). Therefore, (3.4.10) applies to the coset reduction $S^5 = SU(3)/SU(2)$. In addition, note that rescaling the generators and, thus, also the β^a_λ does not affect the transformation law (3.4.10), whence these coframes still furnish a Riemannian metric on S^5 for every $(\gamma, \delta) \in \mathbb{R}_+ \times (\mathbb{R}\backslash\{0\})$. The infinitesimal versions of these transformations are given by the structure constants $f_{AB}{}^{C} = ad_G(I_A)^C{}_B$. This provides enough access to the finite transformations as to investigate globally non-vanishing differential forms on S^5.

At first, in (3.4.10) there is no structure constant having an index $i \in \{6, 7, 8\}$ and one index equal to 5. By the connectedness of $SU(2)$, $Ad_{SU(3)}(g)$ acts trivially on β^5_λ for every $g \in SU(2)$ and $\lambda \in \Lambda$, whence

$$\eta_{|U_\lambda} \coloneqq -\beta^5_\lambda, \quad \lambda \in \Lambda \tag{3.4.14}$$

is a well-defined, globally non-vanishing 1-form on S^5.

There are global 2-forms as well, which can be constructed in a similar manner. Consider a combination $\beta^a_\lambda \wedge \beta^b_\lambda = \beta^a_\lambda \otimes \beta^b_\lambda - \beta^b_\lambda \otimes \beta^a_\lambda$. The transformation of a tensor product reads

$$\beta^a_\rho \otimes \beta^b_\rho = Ad_{SU(3)}(g^{-1})^a{}_c\; Ad_{SU(3)}(g^{-1})^b{}_d\; \beta^c_\lambda \otimes \beta^d_\lambda. \tag{3.4.15}$$

With $g = \exp(\varepsilon\, I_i)$ the infinitesimal version of this is

$$\beta^a_\rho \otimes \beta^b_\rho = \beta^a_\lambda \otimes \beta^b_\lambda - \varepsilon\Big(f_{ic}{}^{a}\; \beta^c_\lambda \otimes \beta^b_\lambda + f_{id}{}^{b}\; \beta^a_\lambda \otimes \beta^d_\lambda\Big) + \mathcal{O}(\varepsilon^2). \tag{3.4.16}$$

Thus, we obtain the infinitesimal transformation

$$\begin{aligned}
\delta_\varepsilon\big(\beta^1_\rho \wedge \beta^2_\rho + \beta^3_\rho \wedge \beta^4_\rho\big) = &- f_{ia}{}^{1}\; \beta^a_\lambda \wedge \beta^2_\lambda - f_{ib}{}^{2}\; \beta^1_\lambda \wedge \beta^b_\lambda\\
&- f_{ia}{}^{3}\; \beta^a_\lambda \wedge \beta^4_\lambda - f_{ib}{}^{4}\; \beta^3_\lambda \wedge \beta^b_\lambda.
\end{aligned} \tag{3.4.17}$$

Inserting e. g. $i = 6$ yields

$$\begin{aligned}
-\delta_\varepsilon\big(\beta^1_\rho \wedge \beta^2_\rho + \beta^3_\rho \wedge \beta^4_\rho\big) = &\; f_{63}{}^{1}\; \beta^3_\lambda \wedge \beta^2_\lambda + f_{64}{}^{2}\; \beta^1_\lambda \wedge \beta^4_\lambda\\
&+ f_{61}{}^{3}\; \beta^1_\lambda \wedge \beta^4_\lambda + f_{62}{}^{4}\; \beta^3_\lambda \wedge \beta^2_\lambda
\end{aligned}$$

$$= \left(-\frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} \right) \beta^3_\lambda \wedge \beta^2_\lambda + \left(\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} \right) \beta^1_\lambda \wedge \beta^4_\lambda = 0. \tag{3.4.18}$$

The structure constants are such that this is completely analogous for $i = 7, 8$ and permutations of the β^a_λ. This implies that the 2-forms defined via

$$\omega^\alpha = \frac{1}{2} \eta^\alpha{}_{bc} \, \beta^b_\lambda \wedge \beta^c_\lambda \quad \forall \, \alpha = 1, 2, 3, \tag{3.4.19}$$

are globally well-defined and nowhere-vanishing. From now on we suppress the index λ of the covering of S^5. Since (η, ω^α) are globally well-defined and there exist local coframes on S^5 around every $x \in M$ such that these forms have components as in (3.2.4), we arrive at

Corollary 3.4.4: *The tuple $(\eta, \omega^1, \omega^2, \omega^3)$ defines an $SU(2)$-structure on S^5 for all values $(\gamma, \delta) \in \mathbb{R}_+ \times (\mathbb{R} \backslash \{0\})$. That is, we have constructed a two-parameter family of $SU(2)$-structures on S^5.*

The $SU(2)$ structures induced for different values of the parameters γ and δ are indeed inequivalent, since the $SU(2)$-transformations on the forms cannot generate the rescalings considered in (3.4.13).

We now investigate whether there is a subset of this two-parameter family of $SU(2)$-structures on S^5 which even is an example for some of the special geometric structures introduced in sections 3.2 and 3.3. To this end, we compute

$$\mathrm{d}\eta = -\,\mathrm{d}(\sigma^* \, \theta^5) = -\,\sigma^* \, (\mathrm{d}\theta^5) = \frac{1}{2} \, f_{AB}{}^5 \, \beta^A \wedge \beta^B = -\frac{\gamma}{2\,\delta^2} \, \omega_3. \tag{3.4.20}$$

For the following computations we use the notation $\beta^{A_1 \dots A_N} = \beta^{A_1} \wedge \dots \wedge \beta^{A_N}$.

$$\begin{aligned}
\mathrm{d}\omega_1 = &\; -\mathrm{d}\beta^1 \wedge \beta^4 + \beta^1 \wedge \mathrm{d}\beta^4 + \mathrm{d}\beta^2 \wedge \beta^3 - \beta^2 \wedge \mathrm{d}\beta^3 \\
= &\; - f_{25}{}^1 \, \beta^{254} - f_{28}{}^1 \, \beta^{284} - f_{36}{}^1 \, \beta^{364} + f_{53}{}^4 \, \beta^{153} \\
&\; + f_{62}{}^4 \, \beta^{162} + f_{38}{}^4 \, \beta^{138} - f_{51}{}^2 \, \beta^{513} - f_{46}{}^2 \, \beta^{463} \\
&\; - f_{81}{}^2 \, \beta^{813} + f_{45}{}^3 \, \beta^{245} + f_{61}{}^3 \, \beta^{261} + f_{84}{}^3 \, \beta^{284} \\
= &\; (f_{25}{}^1 + f_{45}{}^3) \, \beta^{245} + (f_{28}{}^1 + f_{48}{}^3) \, \beta^{248} + (f_{36}{}^1 + f_{64}{}^2) \, \beta^{346} \\
&\; + (f_{35}{}^4 + f_{15}{}^2) \, \beta^{135} + (f_{26}{}^4 + f_{61}{}^3) \, \beta^{126} + (f_{38}{}^4 + f_{18}{}^2) \, \beta^{138} \\
= &\; \frac{1}{\gamma} \, (-\beta^{524} + \beta^{513})
\end{aligned} \tag{3.4.21}$$

$$= \frac{1}{\gamma}\,\eta \wedge \omega_2.$$

Analogously,

$$\begin{aligned}
\mathrm{d}\omega_2 =\; & -\mathrm{d}\beta^1 \wedge \beta^3 + \beta^1 \wedge \mathrm{d}\beta^3 + \mathrm{d}\beta^2 \wedge \beta^4 - \beta^2 \wedge \mathrm{d}\beta^4 \\
=\; & f_{25}{}^1\,\beta^{253} + f_{28}{}^1\,\beta^{283} + f_{74}{}^1\,\beta^{743} - f_{45}{}^3\,\beta^{145} \\
& - f_{72}{}^3\,\beta^{172} - f_{84}{}^3\,\beta^{184} - f_{51}{}^2\,\beta^{514} - f_{37}{}^2\,\beta^{374} \\
& - f_{81}{}^2\,\beta^{814} + f_{53}{}^4\,\beta^{253} + f_{17}{}^4\,\beta^{217} + f_{38}{}^4\,\beta^{238} \\
=\; & (f_{52}{}^1 + f_{35}{}^4)\,\beta^{235} + (f_{82}{}^1 + f_{38}{}^4)\,\beta^{238} + (f_{47}{}^1 + f_{37}{}^2)\,\beta^{347} \\
& + (f_{54}{}^3 + f_{15}{}^2)\,\beta^{145} + (f_{72}{}^3 + f_{71}{}^4)\,\beta^{127} + (f_{84}{}^3 + f_{18}{}^2)\,\beta^{148} \\
=\; & \frac{1}{\gamma}\,(\beta^{523} + \beta^{514}) \\
=\; & -\frac{1}{\gamma}\,\eta \wedge \omega_1.
\end{aligned} \tag{3.4.22}$$

From this we see that the $SU(2)$-structure we are considering is hypo for any values of γ and δ, and nearly hypo precisely for

$$(\gamma, \delta) = \Big(-\frac{1}{3}, \pm\frac{1}{2\sqrt{3}}\Big). \tag{3.4.23}$$

Hence, for these values the $SU(2)$-structure is double hypo, and, as it turns out, it is even Sasaki-Einstein.

It is also interesting to ask which values of the parameters furnish a Sasakian structure on S^5. From the fact that we can always choose a coframe adapted to the $SU(2)$-structure in the sense that $\eta = -\beta^5$ and $\omega^3 = \beta^1 \wedge \beta^2 + \beta^3 \wedge \beta^4$, we can construct ξ and Φ such that (M, g, ξ, η) is an almost metric contact manifold. That is, every $SU(2)$-structure in 5 dimensions is contained in an ambient $U(2)$-structure, similar to what we encountered for $SU(3)$- and $U(3)$-structures in 6 dimensions. Recall from definition 3.3.1 that if in addition $\mathrm{d}\eta = 2\,g(\Phi(\cdot), \cdot)$ holds true, (M, g, ξ, η) is even a metric contact manifold. This is the case for

$$\gamma = -4\,\delta^2. \tag{3.4.24}$$

Of course we still have to check whether or not the metric contact structure is Sasakian. That is, we have to check whether the arising metric contact structures are normal, i.e. whether $\mathcal{L}_\xi\,\Phi = 0$. Recall that

$$\Phi = \beta^1 \otimes e_2 - \beta^2 \otimes e_1 + \beta^3 \otimes e_4 - \beta^4 \otimes e_3. \tag{3.4.25}$$

Also, by applying β^A to $[e_B, e_C]$ we obtain $[e_A, e_B] = f_{AB}{}^C\, e_C$. Additionally employing that on $\Omega^*(M)$ the Lie derivative satisfies

$$\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X \quad \forall\, X \in \Gamma(TM), \tag{3.4.26}$$

we compute

$$\begin{aligned}
\mathcal{L}_{e_5}\, \Phi &= \iota_{e_5}(\mathrm{d}\beta^1) \otimes e_2 + \beta^1 \otimes [e_5, e_2] - \iota_{e_5}(\mathrm{d}\beta^2) \otimes e_1 - \beta^2 \otimes [e_5, e_1] \\
&\qquad + \iota_{e_5}(\mathrm{d}\beta^3) \otimes e_4 + \beta^3 \otimes [e_5, e_4] - \iota_{e_5}(\mathrm{d}\beta^4) \otimes e_3 - \beta^4 \otimes [e_5, e_3] \\
&= \, - (f_{52}{}^1 + f_{51}{}^2)\, \beta^2 \otimes e_2 + (f_{52}{}^1 + f_{51}{}^2)\, \beta^1 \otimes e_1 \\
&\qquad - (f_{54}{}^3 + f_{53}{}^4)\, \beta^4 \otimes e_4 + (f_{54}{}^3 + f_{53}{}^4)\, \beta^3 \otimes e_3 \\
&= 0.
\end{aligned} \tag{3.4.27}$$

While (g, η, Φ, ξ) is a normal metric almost contact structure for arbitrary choice of γ and δ, it is a metric contact structure only for $\gamma = -4\,\delta^2$. Since the metric contact structures obtained this way are even normal, this yields one-parameter family of Sasakian structures on S^5.

In summary, we have constructed a two-parameter family of hypo $SU(2)$-structures on $S^5 = SU(3)/SU(2)$. Each of these gives rise to a normal metric almost contact structure on S^5. Within this family, there lies a one-parameter family of Sasakian structures given by $\gamma = -4\,\delta^2$. This subfamily is Sasaki-Einstein for $(\gamma, \delta) = (-\frac{1}{3}, \pm\frac{1}{2\sqrt{3}})$, which is also the only choice of the parameters that makes the $SU(2)$-structure nearly hypo and, therefore, double hypo.
However, the coset construction of S^5 as considered here allows to make some of the geometric structure carried by the 5-sphere explicit, and to get a grasp of the interplay of the different geometric structures introduced in this chapter.

Chapter 4

Geometry of Instantons

4.1 G-Structures and Instanton Conditions

This section is devoted to the interplay of metric G-structures and gauge theories. We follow [2, 12] in this and the subsequent section. However, we fill in the details, which will enable us to prove the statements of section 4.3. These in turn will have valuable applications in the construction of instanton solutions in chapter 6.

In addition to the frame bundle of M we now consider an arbitrary principal H-bundle $(\mathfrak{B}, \pi, M, H)$ over M. This is the principal bundle of a gauge theory on M. Assume there exists a metric G-structure on M, i. e. a reduction of the bundle $F(TM)$ to a principal subbundle $\mathcal{Q} \subset F(TM)$ with structure group

$$G \subset SO(p,q) \subset GL(D,\mathbb{R}). \tag{4.1.1}$$

We can, thus, always construct a (semi-)Riemannian metric g on M such that the local frames adapted to $\mathcal{Q}$ are orthonormal frames for g. Hence, $\mathcal{Q}$ is contained in the ambient principal $SO(p,q)$-bundle $SO(M,g)$. The existence of a metric G-structure on M does, of course, not reduce the set of possible connections on any gauge principal bundle $(\mathfrak{B}, \pi, M, H)$. However, we will see that one point where the two geometric structures given by a connection on $\mathfrak{B}$ and a metric G-structure on M touch is $\Lambda^2 T^*M$.

The geometry of the instanton equation is footed on the existence of a special isomorphism of vector bundles over semi-Riemannian manifolds. This is yet of purely algebraical origin, as we will explain in the following.

Let us fix some notation first. Consider $\mathbb{R}^D$ endowed with a non-degenerate inner product γ of signature (p,q), $p+q=D$. Be $(v_i)_{i=1,\dots,D}$ an orthonormal basis and $(\theta^i)_{i=1,\dots,D}$ its dual basis of $(\mathbb{R}^D)^*$. We put

$$\begin{aligned} v^i &\coloneqq \sharp_\gamma(\theta^i) = \gamma^{ij}\, v_j \in \mathbb{R}^D, \\ \theta_i &\coloneqq \flat_\gamma(v_i) = \gamma_{ij}\, \theta^j \in (\mathbb{R}^D)^*. \end{aligned} \tag{4.1.2}$$

Embedded into $GL(\mathbb{R}^D)$, a basis of $\mathfrak{so}(p,q)$ is given by (cf. [34])

$$\{E^{ij} = \theta^i \otimes v^j - \theta^j \otimes v^i \,|\, 1 \leqslant i < j \leqslant D\}. \tag{4.1.3}$$

That is, with respect to the basis $(v_i)_{i=1,\dots,D}$ of $\mathbb{R}^D$ the E^{ij} have components

$$(E^{ij})^l{}_k = (\delta^i_k\, \gamma^{jl} - \delta^j_k\, \gamma^{il}). \tag{4.1.4}$$

Also, note that if $\rho : G \to GL(V)$ is a representation of a Lie group G on a finite-dimensional real vector space V, the representation of G on the dual vector space V^* is the inverse transposed representation

$$\rho^{-T} : G \to GL(V^*),\ \rho^{-T}(g) = (\rho(g)^{-1})^T = (\rho(g)^T)^{-1}, \tag{4.1.5}$$

and we have

$$\rho(h)\,(v_i) = \rho(h)^j{}_i\, v_j \quad \text{and} \quad \rho^{-T}(h)\,(\theta^i) = \big(\rho(h^{-1})\big)^T\,(\theta^i) = \rho(h^{-1})^i{}_j\,\theta^j. \tag{4.1.6}$$

The fundamental statement here is the following lemma:

Lemma 4.1.1: *In the above situation, the isomorphism of vector spaces*

$$I_\gamma : \mathfrak{so}(p,q) \to \Lambda^2(\mathbb{R}^D)^*,\ I_\gamma\Big(\frac{1}{2}\,\omega_{ij}\,E^{ij}\Big) = \frac{1}{2}\,\omega_{ij}\,\theta^i \wedge \theta^j. \tag{4.1.7}$$

is an intertwiner of the adjoint representation of $SO(p,q)$ and the representation $\rho^{-T} \otimes \rho^{-T}$ of $SO(p,q)$ on $\Lambda^2(\mathbb{R}^D)^*$.

Proof. Be $h \in SO(p,q)$ and $\rho(h)^i{}_j = h^i{}_j$ the standard representation on $\mathbb{R}^D$. From the fact that $SO(p,q)$ preserves γ we obtain

$$\big(\rho(h^{-1})\big)^i{}_j = (h^{-1})^i{}_j = \gamma_{jk}\,h^k{}_l\,\gamma^{li} = h_j{}^i. \tag{4.1.8}$$

That is,

$$\rho^{-T}(h)\,(\theta^i) = h_j{}^i\,\theta^j. \tag{4.1.9}$$

Since group multiplication on $SO(p,q)$ coincides with matrix multiplication, the adjoint action can be written as

$$\begin{aligned}
(Ad(h)(E^{ij}))^a{}_b &= (h\,E^{ij}\,h^{-1})^a{}_b \\
&= h^a{}_l\,(\delta^i_k\,\gamma^{jl} - \delta^j_k\,\gamma^{il})\,h_b{}^k \\
&= h^{aj}\,h_b{}^i - h^{ai}\,h_b{}^j \\
&= h_n{}^i\,h_m{}^j\,(\delta^n_b\,\gamma^{ma} - \delta^m_b\,\gamma^{na}) \\
&= (h_n{}^i\,h_m{}^j\,E^{nm})^a{}_b \\
&= \Big(\big(\rho(h^{-1})\big)^i{}_n\,\big(\rho(h^{-1})\big)^j{}_m\,E^{nm}\Big)^a{}_b\,.
\end{aligned} \tag{4.1.10}$$

Note that the $SO(p,q)$ property (4.1.8) is crucial here, i. e. this does not extend to any larger subgroup of $GL(D,\mathbb{R})$ (actually, the property (4.1.8) holds for $O(p,q)$, but we also need the special form of all the generators of the Lie algebra (4.1.3), as we used it in the above computation). Thus, putting $E^{ji} = -E^{ij}$ for $j \geqslant i$,

$$\begin{aligned} I_\gamma\Big(Ad(h)\Big(\frac{1}{2}\,\omega_{ij}\,E^{ij}\Big)\Big) &= I_\gamma\Big(\frac{1}{2}\,\omega_{ij}\,h_n{}^i\,h_m{}^j\,E^{nm}\Big) \\ &= \frac{1}{2}\,\omega_{ij}\,h_n{}^i\,h_m{}^j\,\theta^n \wedge \theta^m \\ &= \frac{1}{2}\,\omega_{ij}\,\rho(h)^{-T}(\theta^i) \wedge \rho(h)^{-T}(\theta^j) \\ &= \Big(\rho(h)^{-T} \otimes \rho(h)^{-T}\Big)\Big(\frac{1}{2}\,\omega_{ij}\,\theta^i \wedge \theta^j\Big). \end{aligned} \tag{4.1.11}$$

This shows that I_γ is an intertwiner as stated. □

This algebraic assertion has an important consequence for the geometry on M.

Corollary 4.1.2: *On any semi-Riemannian manifold (M^D, g) the intertwiner I_γ induces an isomorphism of associated vector bundles in the following manner.*
Let $e \in \Gamma_U(SO(M,g))$ be a local orthonormal frame of TM with dual local coframe β. With notation as above, there is an isomorphism of vector bundles defined by

$$I_g : Ad(SO(M,g)) \to \Lambda^2 T^*M,\;\; I_g\big([e,\omega]_{Ad}\big) = [e, I_\gamma(\omega)]_{\Lambda^2}\;. \tag{4.1.12}$$

That is,

$$I_g\bigg(\Big[e,\;\frac{1}{2}\,\omega_{ij}\,E^{ij}\Big]_{Ad}\bigg) = \Big[e,\;\frac{1}{2}\,\omega_{ij}\,\theta^i \wedge \theta^j\Big]_{\Lambda^2} = \frac{1}{2}\,\omega_{ij}\,\beta^i \wedge \beta^j. \tag{4.1.13}$$

Proof. We have to show that the prescription (4.1.12) yields a globally well-defined isomorphism of the two vector bundles.
First, linearity of I_g follows directly from the linearity of I_γ. Also, I_g projects down to the identity on M, whence for every local orthonormal frame e the local representation (4.1.12) of I_g is smooth.
It remains to be shown that these local representations of the action of I_g form a globally well-defined map. Consider another local orthonormal frame $\tilde{e}$ such that e and $\tilde{e}$ are defined on some mutual open subset U of M. There is a smooth transition map $h : U \to SO(p,q)$, $x \mapsto h(x)$ such that $\tilde{e}_x = R_{h(x)}\,e(x)\;\forall\, x \in U$. By the definition of associated vector bundles,

$$\big[e,\,\omega\big]_{Ad} = \big[R_h\,e,\;Ad(h^{-1})(\omega)\big]_{Ad}\;. \tag{4.1.14}$$

We check whether the image under I_g is independent of the representative of $[e,\omega]$:

$$\begin{aligned} I_g\Big(\big[\tilde{e},\, Ad(h^{-1})(\omega)\big]_{Ad}\Big) &:= \big[R_h\, e,\, I_\gamma\big(Ad(h^{-1})(\omega)\big)\big]_{\Lambda^2} \\ &= \Big[R_h\, e,\, \Big(\rho^{-T}(h^{-1})\otimes\rho^{-T}(h^{-1})\Big)\big(I_\gamma(\omega)\big)\Big]_{\Lambda^2} \\ &= [e, I_\gamma(\omega)]_{\Lambda^2}\ . \end{aligned} \tag{4.1.15}$$

Thus, I_g is defined independently of the local orthonormal frame and, therefore, constitutes a globally well-defined isomorphism of associated vector bundles. □

Hence, on every semi-Riemannian manifold there exists this intrinsic link of the adjoint bundle of $SO(M,g)$ and the bundle Λ^2T^*M.

By restriction of I_g to $Ad(\mathcal{Q})$ we obtain a subbundle of Λ^2T^*M of rank $dim(\mathfrak{g})$. This is the point where the geometries of a G-structure and a gauge principal bundle on M are connected. One simply requires that the field strength $F_A \in \Gamma(\Lambda^2T^*M\otimes Ad(\mathfrak{B}))$ be a section of $I_g(Ad(\mathcal{Q}))\otimes Ad(\mathfrak{B})$. That is, the 2-form part of F_A is required to take values in $I_g(Ad(\mathcal{Q}))$ only, instead of in the whole bundle Λ^2T^*M. This motivates the following definitions.

Definition 4.1.3: *Consider a metric G-structure $\mathcal{Q}$ being compatible with a semi-Riemannian metric g on M, and let $(\mathfrak{B},\pi,M,H)$ be a principal fiber bundle. We call*

$$W(\mathcal{Q}) := I_g\big(Ad(\mathcal{Q})\big) \subset \Lambda^2T^*M \tag{4.1.16}$$

the ***instanton bundle*** *of $\mathcal{Q}$.*
A connection $A \in \mathcal{C}(\mathfrak{B})$ is called an ***instanton for $\mathcal{Q}$****, or a* ***$\mathcal{Q}$-instanton****, if*

$$F_A \in \Gamma\big(W(\mathcal{Q})\otimes Ad(\mathfrak{B})\big) \subset \Gamma\big(\Lambda^2(M)\otimes Ad(\mathfrak{B})\big). \tag{4.1.17}$$

Let us finish this section with a side remark on the geometry of adjoint bundles. The fibers of the adjoint bundle $Ad(\mathcal{P})$ of an arbitrary principal bundle $(\mathcal{P},\pi,M,H)$ are endowed with a Lie-algebra structure given by

$$[\cdot,\cdot] : Ad(\mathcal{P})\times Ad(\mathcal{P}) \to Ad(\mathcal{P}),\ \Big[[p,\xi],\,[p,\zeta]\Big]_{Ad(\mathcal{P})} := \big[p,\,[\xi,\zeta]_{\mathfrak{h}}\big]_{Ad(\mathcal{P})}. \tag{4.1.18}$$

Here, $[\xi,\zeta]_{\mathfrak{h}}$ is the commutator of two elements of $\mathfrak{h}$. We have to check that this prescription does indeed yield a well-defined map, i. e. that it is independent of the representatives of the equivalence classes. To this end, denote by $\varphi(\xi)\in\Gamma(TH)$ the left-invariant vector field on H generated by ξ, and the Lie bracket on $\Gamma(TH)$ by $[\cdot,\cdot]_{\Gamma(TH)}$. We compute

$$ad\big(Ad(h^{-1})(\xi)\big)\big(Ad(h^{-1})(\zeta)\big) = \Big[\varphi\big(\big(Ad(h^{-1})(\xi)\big),\, \varphi\big(Ad(h^{-1})(\zeta)\big)\Big]_{\Gamma(TH)\,|e}$$

$$
\begin{aligned}
&= \Big(R_{h*} \left[\varphi(\xi),\, \varphi(\zeta) \right]_{\Gamma(TH)} \Big)_{|e} \\
&= \big(R_{h*}\, \varphi\big([\xi, \zeta]_{\mathfrak{h}} \big) \big)_{|e} \qquad (4.1.19) \\
&= \varphi\Big(Ad(h^{-1})\big([\xi, \zeta]_{\mathfrak{h}} \big) \Big)_{|e} \\
&= Ad(h^{-1})\big([\xi, \zeta]_{\mathfrak{h}} \big).
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\Big[\big[R_h\, p, Ad(h^{-1})(\xi) \big],\, \big[R_h\, p, Ad(h^{-1})(\zeta) \big] \Big]_{Ad(\mathcal{P})} &= \big[R_h\, p, Ad(h^{-1})\big([\xi, \zeta]_{\mathfrak{h}} \big) \big]_{Ad(\mathcal{P})} \\
&= \big[p,\, [\xi, \zeta]_{\mathfrak{h}} \big]_{Ad(\mathcal{P})}\,, \qquad (4.1.20)
\end{aligned}
$$

whence the definition of the bracket on $Ad(\mathcal{P})$ is independent of the representatives. Due to the linearity of the Lie bracket on $\mathfrak{h}$, this yields a $C^\infty(M)$-linear Lie bracket on $\Gamma(Ad(\mathcal{P}))$ and, hence, induces a Lie-algebra structure on the $C^\infty(M)$-module $\Gamma(Ad(\mathcal{P}))$.
This Lie algebroid structure has been noticed before (see e. g. [35]), but in the case of $\mathcal{P} = SO(M, g)$ for some (semi-)Riemannian metric g on M, it is mediated to $\Lambda^2 T^*M$ and $\Omega^2(M)$ via I_g. Note that I_g is $C^\infty(M)$-linear when viewed as a map $I_g : \Gamma(Ad(SO(M, g)) \to \Omega^2(M)$ of $C^\infty(M)$-modules, implying that that the bracket on $\Lambda^2 T^*M$ is $C^\infty(M)$-linear as well. The induced Lie bracket simply reads

$$
\Big[\frac{1}{2}\, \omega_{ij}\, \beta^i \wedge \beta^j,\, \frac{1}{2}\, \mu_{kl}\, \beta^k \wedge \beta^l \Big]_{\Lambda^2 T^*M} = \frac{1}{2} \Big(\sum_{k=1}^{D} \omega_{ik}\, \mu_{kj} \Big) \beta^i \wedge \beta^j. \qquad (4.1.21)
$$

$\Gamma(W(\mathcal{Q}))$ then is a Lie subalgebra of $\Omega^2(M)$ that is induced by Lie subalgebras isomorphic to $\mathfrak{h}$ on the fibers of $\Omega^2(M)$. Moreover, this structure is a Lie algebroid (for the definition see e. g. [36]) over M modeled on either $Ad(\mathcal{P})$ or $\Lambda^2 T^*M$ if taken together with the trivial anchor map.

4.2 Implementations of the Instanton Condition

As we have established above, the instanton condition is the requirement that the 2-form part of the field strength of a connection lies in a certain subbundle of $\Lambda^2 T^*M$. In order to check this for a given connection on a particular geometry, one has to identify this subbundle of $\Lambda^2 T^*M$ and investigate whether the 2-form part of the field strength everywhere is a linear combination of a local basis of this bundle.
This is a problem of pure linear algebra, and further geometric features of the instanton condition are by far not obvious in this formulation. For example, there seems to be no apparent relation to the Yang-Mills equation (with torsion) for the gauge

connection, which we would expect a proper instanton condition to imply. Thus, we are interested in alternative ways to impose the instanton condition that make features like this more manifest. In particular, we should clarify the relation indicated in section 1.3 between the instanton condition of section 4.1 and the gaugino equation

$$\gamma(F^A)(\epsilon) = 0, \tag{4.2.1}$$

which is the requirement that last line of (1.3.1) vanishes. Again we mostly follow [2, 12] in this exposition.

First, for a G-structure with simple Lie algebra $\mathfrak{g}$ and G a proper subgroup of $SO(D)$, one can construct a nowhere-vanishing 4-form on M by mapping the quadratic Casimir of $\mathfrak{g}$ to $\mathfrak{g} \otimes \mathfrak{g}$ via the Killing metric, transferring this to $\Omega^2(M) \otimes \Omega^2(M)$ by means of I_g and then antisymmetrizing the result. This yields a 4-form with G-invariant components on M. One can then show

Proposition 4.2.1: *Let (M^D, g) be a Riemmanian manifold, $G \subset SO(D)$ a proper subgroup with simple Lie algebra $\mathfrak{g}$, $(\mathcal{Q}, \pi, M, G)$ a G-structure on M compatible with g and $Q \in \Omega^4(M)$ the 4-form constructed via the above prescription.*
Then we have for all $\mu \in \Omega^2(M)$ that

$$\mu \in \Gamma(W(\mathcal{Q})) \quad \Leftrightarrow \quad *(*Q \wedge \mu) = -\mu, \tag{4.2.2}$$

where $*$ is the Hodge star induced by g.

This version of the instanton condition has first been introduced in [37]. If we take a connection $A \in \mathcal{C}(\mathfrak{B})$ on $(\mathfrak{B}, \pi, M, H)$ solving (4.2.2), we see upon applying the covariant differential d_A to this identity that

$$\mathrm{d}_A{*F_A} = -(\mathrm{d}{*Q}) \wedge F_A. \tag{4.2.3}$$

That is, the instanton equation (4.2.2) implies the so-called Yang-Mills equation with torsion (4.2.3). Clearly, (4.2.2) implies the Yang-Mills equation without torsion whenever Q is co-closed, which is the case for integrable G-structures, i. e. on manifolds with special holonomy [2].
On a generic $SU(2)$ 5-manifold, the only nowhere-vanishing 1-forms which are present generically are $\phi\eta$ for nowhere-vanishing function $\phi \in C^\infty(M)$. The linear map

$$\mu \mapsto *(\eta \wedge \mu) \tag{4.2.4}$$

has eigenspaces and eigenvalues as required in [12], whence η must be the $(D-4)$-form arising from the above construction on such spaces. That is, there we have $*Q = \eta$. Similarly, on generic 6-dimensional $SU(3)$-manifolds we have $*Q = \omega$.

This is because Q is constructed by purely algebraic means, insensible to any geometric properties of the G-structure, and the only 4-forms present on any generic $SU(3)$-structure 6-manifold are $*(\phi\,\omega)$. Again, ω yields the desired eigenspaces and eigenvalues.
However, we could in principle generalize this second version of the instanton condition by using different nowhere-vanishing $(D-4)$-forms that may occur in certain situations. We will encounter such a case in section 5.4.

Another important case is given when the preimage of G under the double covering $\lambda : Spin(D) \to SO(D)$ stabilizes a spinor in the spinor representation ρ_s of $Spin(D)$, such that there exists a nowhere-vanishing spinor $\epsilon \in \Gamma(\mathcal{S}(M,g))$ on M. Then we have

Proposition 4.2.2: *Let (M^D, g) be a Riemannian manifold, $G \subset SO(D)$ as described above, and let $(\mathcal{Q}, \pi, M, G)$ be a G-structure on M defined by a nowhere-vanishing spinor $\epsilon \in \Gamma(\mathcal{S}(M,g))$ on M.*
Then for $\mu \in \Omega^2(M)$ we have that

$$\mu \in \Gamma(W(\mathcal{Q})) \quad \Leftrightarrow \quad \gamma(\mu)(\epsilon) = 0. \tag{4.2.5}$$

This is due to the fact that $\gamma(\beta^i)\,\gamma(\beta^j) = 2\,\rho_s(E^{ij})$ on spinors of $Spin(D)$ [34]. As has been worked out in [2], the condition (4.2.5) implies the Yang-Mills equation without torsion if the spinor ϵ is Killing with respect to the Levi-Civita connection of the metric g.
The last version of the instanton condition brings full circle the detour into the geometry of G-structures and instantons that we embarked on motivated by the gaugino equation (4.2.1). Recall that this is one of the supersymmetry conditions of heterotic supergravity (1.3.1).

These two versions of the instanton condition are much more explicit than its first version (4.1.17). In this thesis we will mostly make use of (4.1.17) and (4.2.2), since we will consider G-structures that either do not have a Killing spinor, or whose Killing spinor is more difficult to handle than the generic one of e. g. a Sasaki-Einstein $SU(2)$-structure in 5 dimensions.

4.3 Deformation of G-Structures

In this section we investigate the geometry of G-structures and deformations thereof on a generic manifold M. The considerations in this chapter seem to be new to the literature. They will have important implications on the instantons defined by G-structures related by a special class of deformations.

Let M be a manifold of dimension D and $F(TM)$ its frame bundle. As additional data, assume that there is a G-structure $\mathcal{Q} \subset F(TM)$ on M with structure group $G \subset GL(D, \mathbb{R})$. This G-structure may be defined by defining sections

$$\tau \in \Gamma(E), \quad \text{where} \quad E = F(TM) \times_{(GL(D,\mathbb{R}),\rho)} V \tag{4.3.1}$$

for representations ρ of $GL(D, \mathbb{R})$ on a vector space V. In this case, G is defined as the stabilizer of an element $\tau_0 \in V$ with respect to ρ, and for every $x \in M$ there is a $q \in \mathcal{Q}_x$ such that

$$\tau_x = [q, \tau_0], \tag{4.3.2}$$

as we introduced in section 2.1. Recall from there the crucial result that for associated bundles we have

$$E = F(TM) \times_{(GL(D,\mathbb{R}),\rho)} V = \mathcal{Q} \times_{(G,\rho_{|G})} V. \tag{4.3.3}$$

As in chapter 2, in our abstract considerations we use only one defining section as a representative for all the defining sections of $\mathcal{Q}$. In general, G-structures have several defining sections simultaneously, as for instance $SU(2)$-structures in dimension 5 do (cf. section 3.2).

Now we deform the G-structure $\mathcal{Q}$. To this end, consider a map

$$h : M \to GL(D, \mathbb{R}), \; x \mapsto h(x). \tag{4.3.4}$$

This induces a map from the principal bundle $F(TM)$ to itself, given by the prescription $e \mapsto R_{h(\pi(e))}\, e$. The question we are interested in now is, whether or not the image of $\mathcal{Q}$ under this map again is a G-structure. Recall that we view G as a fixed subgroup of $GL(D, \mathbb{R})$. The answer is then given by the following proposition:

Proposition 4.3.1: *Let $\mathcal{Q} \subset F(TM)$ be a G-structure on M, and consider a map $h : M \to GL(D, \mathbb{R})$, $x \mapsto h(X)$. Denote the image of $\mathcal{Q}$ under the right-action of h by $\mathcal{Q}' \coloneqq R_h\, \mathcal{Q}$. That is,*

$$\mathcal{Q}' = \left\{ R_{h(\pi(q))}\, q \,|\, q \in \mathcal{Q} \right\}. \tag{4.3.5}$$

Then the following statements are equivalent:

(1) $\mathcal{Q}'$ is a G-structure on M.

(2) h takes values in $N_{GL(D,\mathbb{R})}(G)$ only, where

$$N_{GL(D,\mathbb{R})}(G) \coloneqq \left\{ a \in GL(D, \mathbb{R}) \,|\, a\, g\, a^{-1} \in G \;\; \forall\, g \in G \right\} \tag{4.3.6}$$

is the normalizer of G in $GL(D, \mathbb{R})$.

(3) If $\mathcal{Q}$ has a defining section $\tau \in \Gamma(E)$ in a vector bundle associated to $F(TM)$ as above, the prescription

$$\tau'_{\pi(q)} \coloneqq [R_{h(\pi(q))}\, q,\, \tau_0] \tag{4.3.7}$$

yields a globally well-defined section of E.

Proof. (1) $\Rightarrow$ (2): Assume that $\mathcal{Q}' = R_h\,\mathcal{Q} \subset F(TM)$ is a G structure on M. In particular, $\mathcal{Q}'$ is a principal subbundle of $F(TM)$ and it carries a right-action of G given by the restriction of the right-action existing on $F(TM)$. In this case, $\mathcal{Q}'$ is closed under this right-action of G, i. e.

$$R_g\,q' \in \mathcal{Q}' \quad \forall\, g \in G,\, q' \in \mathcal{Q}'. \tag{4.3.8}$$

Now for every $q' \in \mathcal{Q}'$ there is a unique $q \in \mathcal{Q}$ such that $q' = R_{h(x)}\,q$, where $x = \pi(q')$. Hence,

$$R_g\,q' = R_g\,R_{h(x)}\,q = R_{h(x)\,g}\,q = R_{h(x)}\,R_{\alpha(h(x))(g)}\,q \tag{4.3.9}$$

must still lie in $\mathcal{Q}'$. Here we denoted the inner automorphism of H by

$$\alpha(h_1)(h_2) \coloneqq h_1\,h_2\,h_1^{-1}. \tag{4.3.10}$$

Since $\mathcal{Q}'$ is defined as the image of $\mathcal{Q}$ under R_h, we have $R_g\,q' \in \mathcal{Q}'$ if and only if

$$R_{\alpha(h(x))(g)}\,q \in \mathcal{Q}. \tag{4.3.11}$$

This is equivalent to

$$\alpha(h(x))(g) \in G \quad \forall\, x \in M, \tag{4.3.12}$$

because $\mathcal{Q}$ is closed under the right-action of G, and because the right-action of G on $\mathcal{Q}$ is simply transitive. But this is just the condition that $h(x) \in N_{GL(D,\mathbb{R})}(G)$ for all $x \in M$.

(2) $\Rightarrow$ (1): First, note that, as R_h is a diffeomorphism on $F(TM)$, $\mathcal{Q}'$ endowed with the induced differentiable structure is indeed a submanifold of $F(TM)$. If this was to be a G-structure on M, the restriction of the right-action of $GL(D,\mathbb{R})$ on $F(TM)$ should be a right-action on $\mathcal{Q}'$ as well. As before, the right action is given by

$$R_g\,q' = R_g\,R_{h(x)}\,q = R_{h(x)\,g}\,q = R_{h(x)}\,R_{\alpha(h(x))(g)}\,q, \tag{4.3.13}$$

which we know to be an element of $\mathcal{Q}'$ since we assume statement (2) of the proposition. Furthermore, as the right-action of G on $\mathcal{Q}$ is simply-transitive and $\alpha(h(x))$ is an automorphism of G for every $x \in M$, the above G-action on $\mathcal{Q}'$ is simply transitive. Hence, $\mathcal{Q}'$ is a principal G-subbundle of $F(TM)$, and thus, by definition, a G-structure on M.

(2) $\Leftrightarrow$ (3): In the case of (2) we have

$$\tau'_{\pi(q)} \coloneqq [R_{h(x)}\,q,\, \tau_0] = [q,\, \rho(h(x))(\tau_0)]. \tag{4.3.14}$$

For $\tau'_{\pi(q)}$ to be well-defined, this has to be independent of the particular choice of q. Thus, we compute

$$[R_{h(x)}\,R_g\,q,\, \tau_0] = [R_{\alpha(h(x)^{-1})(g)}\,R_{h(x)}\,q,\, \tau_0] \tag{4.3.15}$$

$$= [R_{h(x)}\, q,\, \rho(\alpha(h(x)^{-1})(g))(\tau_0)].$$

This is equal to $[R_{h(x)}\, q,\, \tau_0]$ if and only if

$$\rho\big(\alpha(h(x)^{-1})\,(g)\big)(\tau_0) = \tau_0, \tag{4.3.16}$$

which is just the requirement that $\alpha(h(x)^{-1})(g) \in G$. In turn, this is equivalent to $h(x) \in N_{GL(D,\mathbb{R})}(G)$ for all $x \in M$. As h is smooth and defined globally, so is τ'. □

Definition 4.3.2: *We say that the tuple $(\mathcal{Q}, \mathcal{Q}')$ of G-structures on M satisfies the* ***normal deformation property*** *with respect to h if there exists a smooth map $h : M \to N_{GL(D,\mathbb{R})}(G)$, $x \mapsto h(x)$ such that*

$$\mathcal{Q}'_x = R_{h(x)}\, \mathcal{Q}_x. \tag{4.3.17}$$

That is, for every $q' \in \mathcal{Q}'$ there exists a unique $q \in \mathcal{Q}$ such that $q' = R_{h(\pi(q'))}\, q$.

Let us consider the case where the G-structure is defined by a tensor field, as for example a Riemannian metric. The principal bundle $\mathcal{Q}$ of the G-structure in such a case is the set of all frames of TM with respect to which the defining tensor field has certain standard components. To all frames of T_xM belonging to $\mathcal{Q}$ we now apply the transformation

$$e_\mu \mapsto h(x)^\nu{}_\mu\, e_\nu. \tag{4.3.18}$$

We then ask, whether or not there exists a globally well-defined tensor field that takes the very same standard components with respect to the transformed frames, that the original tensor field took with respect to e. This is equivalent to the statement, that the transformed frames constitute a G-structure. It is crucial that this must hold true for *all* bases belonging to the original G-structure $\mathcal{Q}$. As we have just proven formally, this is the case if and only if h takes values in the normalizer of G in $GL(D, \mathbb{R})$ exclusively.
The easiest example where this holds true is

$$h = \phi\, \mathbb{1}_D, \tag{4.3.19}$$

where $\phi \in C^\infty(M)$ is a smooth, nowhere-vanishing function on M. R_h then just rescales the bases adapted to $\mathcal{Q}$, and, as h commutes with all of $GL(D, \mathbb{R})$, this yields a new G-structure $\mathcal{Q}'$ on M.

In proposition 4.3.1 we have found a large class of deformations of that produce new G-structures from given ones. It will also be important that if, conversely, we were given a G-structure $\mathcal{Q}$ on M and a map $h : M \to GL(D, \mathbb{R})$ such that $R_h\, \mathcal{Q}$ is a G-structure again, we can infer that h takes values in the normalizer of G in $GL(D, \mathbb{R})$ exclusively.

The map $R_h : \mathcal{Q} \to \mathcal{Q}'$ is not an isomorphism of principal G-bundles since it does not map the right-actions of G on the respective bundles to one another. Thus, in general, connections on $\mathcal{Q}'$ are not the pullbacks of connections on $\mathcal{Q}$ along R_h. Nevertheless, propositions 2.2.1 and 2.2.2 provide a bijection between $\mathcal{C}(\mathcal{Q})$ and $\mathcal{C}(\mathcal{Q}')$ in a slightly restricted situation.

Proposition 4.3.3: *Assume that $(\mathcal{Q}, \mathcal{Q}')$ are two G-structures on M that satisfy the normal deformation property with respect to h. Furthermore, let the splitting $\mathfrak{gl}(D,\mathbb{R}) = \mathfrak{g} \oplus \mathrm{m}$ be invariant under $Ad_{GL}(G)$. We identify $A \in \mathcal{C}(\mathcal{Q}) \hookrightarrow \mathcal{C}(F(TM))$ with its extension to a connection on $F(TM)$.*
There exists a bijective map

$$f_{\mathcal{Q},h} : \mathcal{C}(\mathcal{Q}) \to \mathcal{C}(\mathcal{Q}'),\ f_{\mathcal{Q},h}(A) = pr_{\mathfrak{g}} \circ A_{|\mathcal{Q}'}. \tag{4.3.20}$$

Proof. Proposition 2.2.2 directly implies that $f_{\mathcal{Q},h}(A) \in \mathcal{C}(\mathcal{Q}')$ is a connection on $\mathcal{Q}'$. In order to prove that this map is bijective, we recall that $\mathcal{C}(\mathcal{Q})$ is affine vector space modeled over $\Omega^1_{hor}(\mathcal{Q},\mathfrak{g})^{(G,Ad)}$. Due to the Ad-equivariance of elements of that vector space, we can extend every $\omega \in \Omega^1_{hor}(\mathcal{Q},\mathfrak{g})^{(G,Ad)}$ to a horizontal, Ad-equivariant 1-form $\omega \in \Omega^1_{hor}(F(TM),\mathfrak{gl}(D,\mathbb{R}))^{(GL(D,\mathbb{R}),Ad)}$. From the proof of proposition 2.2.2 we see that its restriction to $\mathcal{Q}'$ is horizontal and Ad-equivariant. Moreover, it is $\mathfrak{g}$-valued on $\mathcal{Q}'$. To see this, let e be a local section of $\mathcal{Q}$. Then, $e' := R_h \circ e$ is a local section of $\mathcal{Q}'$. We have

$$e'^*\omega = Ad(h^{-1}) \circ e^*\omega. \tag{4.3.21}$$

This is $\mathfrak{g}$-valued because $e^*\omega$ is $\mathfrak{g}$-valued by construction and h takes values in the the normalizer of G.
Therefore, the auxiliary map given by

$$f : \Omega^1_{hor}(\mathcal{Q},\mathfrak{g})^{(G,Ad)} \to \Omega^1_{hor}(\mathcal{Q}',\mathfrak{g})^{(G,Ad)},\ \omega \mapsto \omega_{|\mathcal{Q}'} \tag{4.3.22}$$

is an isomorphism of vector spaces.
Consider a connection $A \in \mathcal{C}(\mathcal{Q})$. Every other connection on $\mathcal{Q}$ is of the form $A + \omega$ for an $\omega \in \Omega^1_{hor}(\mathcal{Q},\mathfrak{g})^{(G,Ad)}$. Upon application of $f_{\mathcal{Q},h}$ we obtain

$$f_{\mathcal{Q},h}(A+\omega) = f_{\mathcal{Q},h}(A) + f(\omega) = f_{\mathcal{Q},h}(A) + \omega_{|\mathcal{Q}'}. \tag{4.3.23}$$

Since $f_{\mathcal{Q},h}(A) \in \mathcal{C}(\mathcal{Q}')$ and f is bijective, we can obtain every connection on $\mathcal{Q}'$ by these means. Thus, $f_{\mathcal{Q},h}$ is surjective.
If $A_1, A_2 \in \mathcal{C}(\mathcal{Q})$, we have $A_1 - A_2 \in \Omega^1_{hor}(\mathcal{Q},\mathfrak{g})^{(G,Ad)}$, and

$$f_{\mathcal{Q},h}(A_1) - f_{\mathcal{Q},h}(A_2) = f_{\mathcal{Q},h}(A_1 - A_2) = f(A_1 - A_2). \tag{4.3.24}$$

Due to the bijectivity of f, this is equal to zero if and only if $A_1 = A_2$, whence $f_{\mathcal{Q},h}$ is injective as well. □

However,

$$f_{\mathcal{Q},h}{}^{-1} \neq f_{R_h\mathcal{Q},h^{-1}}, \tag{4.3.25}$$

which can be seen by using local representations with respect to e and $e' = R_h \circ e$. The reason for this is that the $pr_{\mathfrak{g}}$ in the definition of $f_{\mathcal{Q},h}$ truncates the inhomogeneous term in the transformation of A.
The local representation of $f_{\mathcal{Q},h}(A)$ reads

$$e'^*(f_{\mathcal{Q},h}(A)) = pr_{\mathfrak{g}} \circ \left(Ad(h^{-1}) \circ e^*A + h^*\mu_{GL}\right) \tag{4.3.26}$$

$$= Ad(h^{-1}) \circ e^*A + pr_{\mathfrak{g}} \circ h^*\mu_{GL}. \tag{4.3.27}$$

This bijection of $\mathcal{C}(\mathcal{Q})$ and $\mathcal{C}(\mathcal{Q}')$ does, in particular, apply to normal deformations of metric G-structures, where $\mathfrak{so}(D) = \mathfrak{g} \oplus \mathfrak{n}$ is invariant under $Ad_{SO}(G)$. In order that the proposition applies, we have to make sure that the splitting $\mathfrak{gl}(D,\mathbb{R}) = \mathfrak{g} \oplus \mathrm{m}$ is invariant under $Ad_{GL}(G)$. First, the splitting

$$\mathfrak{gl}(D,\mathbb{R}) = \mathfrak{so}(D) \oplus \mathrm{sym}_0 \oplus \mathrm{tr}, \tag{4.3.28}$$

is invariant under the adjoint action of $SO(D)$. Here sym_0 is the space of symmetric, traceless matrices, and tr is the space of multiples of the unit matrix. The invariance of the splitting of $\mathfrak{so}(D)$ under G and the fact that for metric G-structures we have $G \subset SO(D) \subset GL(D,\mathbb{R})$, imply the invariance of $\mathfrak{gl}(D,\mathbb{R}) = \mathfrak{g} \oplus \mathrm{m}$ under G.
As an example, consider $G = SO(D)$ and $h = \phi\,\mathbb{1}$. The normal deformation leads from the Riemannian metric g to $\phi^2\,g$. Proposition 4.3.3 not only implies that there is a one-to-one correspondence between connections preserving g and $\phi^2\,g$, but also that their local representations are related via (4.3.26). Therefore, once we know a connection A preserving g, we can use both A and $f_{\mathcal{Q},h}(A)$ in computations.
Nevertheless, we will encounter special cases with simpler relations of connections on $\mathcal{Q}$ and $\mathcal{Q}'$ in sections 6.4.2 and 6.6.

Let us now work our way towards the instanton conditions (in the sense of (4.1.17)) induced by the G-structures $\mathcal{Q}$ and $\mathcal{Q}'$ satisfying the normal deformation property. We begin by considering their adjoint bundles.

Lemma 4.3.4: *Assume we are given two G-structures $\mathcal{Q}$ and $\mathcal{Q}'$ on M that satisfy the normal deformation property with respect to $h : M \to GL(D,\mathbb{R})$.*
Then their adjoint bundles coincide, i. e.

$$Ad(\mathcal{Q}) = Ad(\mathcal{Q}') \subset Ad(F(TM)). \tag{4.3.29}$$

Proof. We have

$$Ad(F(TM)) = F(TM) \times_{(GL(D,\mathbb{R}),Ad_{Gl})} \mathfrak{gl}(D,\mathbb{R})$$

$$= \mathcal{Q} \times_{(G, Ad_{Gl\,|G})} \mathfrak{gl}(D, \mathbb{R}) \tag{4.3.30}$$
$$= \mathcal{Q}' \times_{(G, Ad_{Gl\,|G})} \mathfrak{gl}(D, \mathbb{R}).$$

Recall that on $\mathfrak{g}$ the restriction of Ad_{Gl} to G coincides with the adjoint representation of G, for we view G as a fixed subgroup of $GL(D, \mathbb{R})$. Therefore,

$$\begin{aligned} Ad(\mathcal{Q}) &= \mathcal{Q} \times_{(G, Ad_G)} \mathfrak{g} = \mathcal{Q} \times_{(G, Ad_{Gl\,|G})} \mathfrak{g}, \\ Ad(\mathcal{Q}') &= \mathcal{Q}' \times_{(G, Ad_G)} \mathfrak{g} = \mathcal{Q}' \times_{(G, Ad_{Gl\,|G})} \mathfrak{g}, \end{aligned} \tag{4.3.31}$$

and, hence, both are vector *subbundles* of $Ad(F(TM))$. In particular, $Ad(\mathcal{Q})$ is the subbundle of $Ad(F(TM))$ whose elements can be written as $[q, \xi]$ for some $q \in \mathcal{Q}$ and $\xi \in \mathfrak{g}$. For elements of $Ad(\mathcal{Q})$ we have

$$[q,\, \xi] = [R_{h(x)}\, q,\, Ad_{Gl}(h(x)^{-1})(\xi)] = [q',\, Ad_{Gl}(h(x)^{-1})(\xi)]. \tag{4.3.32}$$

As the adjoint action of $h(x)$ preserves G in $GL(D, \mathbb{R})$, on the Lie algebra level it also preserves $\mathfrak{g}$ in $\mathfrak{gl}(D, \mathbb{R})$. Hence,

$$[q,\, \xi] = [q',\, Ad_{Gl}(h(x)^{-1})(\xi)] \in Ad(\mathcal{Q}') \subset Ad(F(TM)). \tag{4.3.33}$$

Thus, $Ad(\mathcal{Q}) \subset Ad(\mathcal{Q}')$, and the same same arguments hold for the converse direction as well. □

Therefore, the adjoint bundles of G-structures that are related by means of a normal deformation coincide. However, this does not directly translate to the induced instanton bundles.

First, we have to make sure that $G \subset SO(D)$, since we need a Riemannian metric compatible with the G-structure in order to even define $W(\mathcal{Q})$. Again, recall that we consider G and $SO(D)$ as fixed subgroups of $GL(D, \mathbb{R})$, where we consider the standard embedding of $SO(D)$ into $GL(D, \mathbb{R})$. Therefore, let us now restrict ourselves to G-structures allowing for a compatible Riemannian metric.

The problem is that the transformations $h : M \to GL(D, \mathbb{R})$ that we consider need not be orthogonal, whence there might be no Riemannian metric compatible with $\mathcal{Q}$ and $\mathcal{Q}'$ simultaneously. Thus, let g be a Riemannian metric compatible with $\mathcal{Q}$ and g' a Riemannian metric compatible with $\mathcal{Q}'$. The instanton bundles are given by

$$W(\mathcal{Q}) = I_g(Ad(\mathcal{Q})) \quad \text{and} \quad W(\mathcal{Q}') = I_{g'}(Ad(\mathcal{Q}')). \tag{4.3.34}$$

First, if $g = g'$, the previous lemma implies that $W(\mathcal{Q}) = W(\mathcal{Q}')$. For the general case, consider an element $\mu \in W(\mathcal{Q})$. With respect to a frame e adapted to $\mathcal{Q}$, this is of the form

$$\mu = \frac{1}{2}\, \mu_{ab}\, \beta^a \wedge \beta^b = I_g\Big(\Big[q,\, \frac{1}{2}\, \mu_{ab}\, E^{ab}\Big]\Big), \tag{4.3.35}$$

where $\frac{1}{2}\,\mu_{ab}\,E^{ab} \in \mathfrak{g} \subset \mathfrak{so}(D) \subset \mathfrak{gl}(D,\mathbb{R})$. Recall that

$$e_a = [e, v_a] \quad \text{and} \quad \beta^a = [e, \theta^a], \tag{4.3.36}$$

using the conventions of section 4.1.
We may express the element μ of $\Lambda^2 T^*M$ with respect to a local frame given by $e' = R_{h(x)}\,e$, which is a local section of $\mathcal{Q}'$. The coframe associated to this frame is

$$(\beta')^a = [R_h \circ e, \theta^a] = [e,\, \rho^{-T}(h)\,(\theta^a)] = [e,\, \rho(h^{-1})^a{}_b\,\theta^b] = \rho(h^{-1})^a{}_b\,\beta^b. \tag{4.3.37}$$

That is, dropping the ρ of the standard representation of $GL(D,\mathbb{R})$ on $\mathbb{R}^D$, we can compute the preimage of μ under $I_{g'}$:

$$\begin{aligned} \mu &= \frac{1}{2}\,\mu_{ab}\,\beta^a \wedge \beta^b = \mu = \frac{1}{2}\,h(x)^a{}_i\;\mu_{ab}\;h(x)^b{}_j\;(\beta')^i \wedge (\beta')^j \\ &= I_{g'}\Big(\Big[e',\,\frac{1}{2}\,h(x)^a{}_i\;\mu_{ab}\,h(x)^b{}_j\;E^{ij}\Big]\Big). \end{aligned} \tag{4.3.38}$$

We are still searching for conditions on h such that the instanton bundles of $\mathcal{Q}$ and $\mathcal{Q}' = R_h\,\mathcal{Q}$ coincide. From the above result one can see that

$$\begin{aligned} \mu \in W(\mathcal{Q}') \quad &\Leftrightarrow \quad \Big[e',\,\frac{1}{2}\,h(x)^a{}_i\;\mu_{ab}\,h(x)^b{}_j\;E^{ij}\Big] \in Ad(\mathcal{Q}') \\ &\Leftrightarrow \quad h(x)^a{}_i\;\mu_{ab}\,h(x)^b{}_j\;E^{ij} \;\in \mathfrak{g}. \end{aligned} \tag{4.3.39}$$

Thus, the question is whether the linear map given by

$$\mu_{ab}\,E^{ab} \mapsto h(x)^a{}_i\;\mu_{ab}\,h(x)^b{}_j\;E^{ij} \tag{4.3.40}$$

preserves $\mathfrak{g} \subset \mathfrak{so}(D)$ for all $x \in M$.
To sum up, we arrive at

Corollary 4.3.5: *The instanton bundles of a pair $(\mathcal{Q}, \mathcal{Q}')$ of G-structures on M satisfying the normal deformation property with respect to $h : M \to GL(D,\mathbb{R})$ coincide if and only if the map*

$$\Phi_{h(x)} : \mathfrak{so}(D) \to \mathfrak{so}(D), \quad \mu_{ab}\,E^{ab} \mapsto h(x)^a{}_i\;\mu_{ab}\,h(x)^b{}_j\;E^{ij} \tag{4.3.41}$$

preserves $\mathfrak{g}$ as a vector subspace of $\mathfrak{so}(D)$ for all $x \in M$.
In particular, this is true whenever h takes values in $SO(D)$, that is, whenever the two G-structures are compatible with the same Riemannian metric.
This also holds true for Φ_h being proportional to the identity transformation on a subspace of $\mathfrak{so}(D)$ containing $\mathfrak{g}$.

Proof. The classification

$$W(\mathcal{Q}) = W(\mathcal{Q}') \quad \Leftrightarrow \quad \Phi_{h(x)}(\mathfrak{g}) \subset \mathfrak{g} \quad \forall\, x \in M \tag{4.3.42}$$

is evident from the arguments that led to the corollary.
If h is $SO(D)$-valued, the action of h in (4.3.41) translates to the adjoint action of $SO(D)$ on $\mathfrak{so}(D)$ by lemma 4.1.1. As we additionally have $h(x) \in N_{SO(D)}(G)$ for all $x \in M$, this preserves the subspace $\mathfrak{g}$ of $\mathfrak{so}(D)$. □

Although this treatment may seem somewhat technical at this point, both of the cases mentioned in the corollary will occur in the constructions in chapter 5 and chapter 6. There we will find these general assertions very useful, since they will provide us with immediate knowledge about the geometries we encounter there as well as about how their instanton bundles are related.

As an example, let us consider a 5-manifold M^5 carrying an $SU(2)$-structure defined by (η, ω^α) (cf. section 3.2). The representation of $SU(2)$ on $\mathbb{R}^5$ that is used to associate the tangent bundle to $\mathcal{Q} \subset SO(M,g)$ splits into a one-dimensional and a four-dimensional representation. This can be seen from the fact that η is a defining section whose dual singles out a 1-dimensional subbundle of TM^5. Since the representations are irreducible, every element of $SU(2) \subset SO(5)$ commutes with matrices that are proportional to the identity on these two invariant subspaces due to Schur's lemma.
Hence, we may consider a deformation of the $SU(2)$-structure induced by the matrix

$$C(\phi) := \begin{bmatrix} \phi^{-1}\, \mathbb{1}_4 & \\ & 1 \end{bmatrix}, \tag{4.3.43}$$

where $\phi \in C^\infty(M, \mathbb{R}_+)$. Under this deformation, the local coframes β of the $SU(2)$-structure transform as

$$\beta^a \mapsto \phi\, \beta^a \quad \text{and} \quad \beta^5 \mapsto \beta^5. \tag{4.3.44}$$

We, therefore, infer the following statement:

Proposition 4.3.6: *Let $\mathcal{Q} \subset SO(M,g)$ be the $SU(2)$-structure defined by (η, ω^α). If the defining sections take their standard form with respect to the local section e of $\mathcal{Q}$, then the forms $(\eta, \phi^2\, \omega^\alpha)$ take their standard forms with respect to the local frame $e' = R_{C(\phi)}\, e$.*
The forms $(\eta, \phi^2\, \omega^\alpha)$ therefore define an $SU(2)$-structure $\mathcal{Q}' \subset SO(M, g_\phi)$, where

$$g_\phi = \eta \otimes \eta + \phi^2\, \delta_{ab}\, \beta^a \otimes \beta^b, \quad a,b = 1,\dots,4. \tag{4.3.45}$$

From the fact the induced deformation of the $SU(2)$-structure again is an $SU(2)$-structure we know by proposition 4.3.1 that C takes values in the normalizer of

$\mathfrak{su}(2)$ in $\mathfrak{gl}(5,\mathbb{R})$. However, it is not $\mathfrak{so}(5)$-valued, as it changes the metric on M^5. Nevertheless, $SU(2)$ is embedded into the upper-left (4×4)-part of $SO(5)$, such that

$$\mathfrak{su}(2) \subset \left\{ \frac{1}{2}\, \mu_{\rho\lambda}\, E^{\rho\lambda} \,\middle|\, \mu_{5\lambda} = -\mu_{\lambda 5} = 0 \right\} \subset \mathfrak{so}(5). \tag{4.3.46}$$

As the upper-left (4×4)-part of C is proportional to $\mathbb{1}_4$, the intermediate subspace above is invariant under $\Phi_{C(\phi(x))}$ for all $x \in M$. Hence, the original and the new G-structure define the same instanton bundles. This particular example has already been used in [2], and we will reproduce the respective results from that paper in section 6.2.
As another example, considering transformations of the form $\phi(x)\,\mathbb{1}_D$ on M^D reproduces the known fact that the instanton condition is invariant under conformal transformations.

Chapter 5

SU(3)- from SU(2)-Structures

5.1 General Statements

After the general considerations of the previous chapters, we now come to more explicit constructions. Here, the general task is to reduce abstract equations on geometric quantities to differential or algebraic equations on functions, which we can solve explicitly.
To this end, we need geometries that are more accessible than just generic geometric spaces. We either need additional geometric data, like group actions or certain global sections of bundles over M, that we can use in explicit constructions and calculations, or, even more, we could use coordinate charts. These would, however, restrict the admissible manifolds to very specific ones.
Conical extensions of generic manifolds provide a class of geometric spaces which benefit from both these features to certain amounts. They have been used successfully in constructions of solutions to instanton and heterotic supergravity equations for example in [2–4]. In differential geometry, cone constructions are a well-established tool for constructions of several special geometries, see for example [17, 24, 38, 39].

We are still interested in constructing 6-dimensional manifolds with $SU(3)$-structure. Consider a 5-dimensional Riemannian manifold (M^5, g) which is endowed with a one-parameter family of $SU(2)$-structures

$$\mathcal{Q}_r \subset F(TM^5),\; r \in I, \tag{5.1.1}$$

where I is an interval. Denote by $\iota_r : M^5 \hookrightarrow M^5 \times I$, $x \mapsto (x, r)$ the embedding of M^5 into $M^5 \times I$ as the slice at parameter value $r \in I$. As the $\mathcal{Q}_r$ are subbundles of $F(TM^5)$, we can construct embeddings

$$\mathcal{Q}_r \mapsto \hat{\mathcal{Q}}_r,\; (e_{r,\mu}) \mapsto \big(\iota_{r*}(e_{r,\mu}),\, e_6\big) \in F\big(T(M^5 \times I)\big) \tag{5.1.2}$$

of $\mathcal{Q}_r$ into $F(TM^5 \times I)$, where e_r is a local section of $\mathcal{Q}_r$ and $e_6 = \partial_r$. We now require the family $\{\mathcal{Q}_r\}_{r\in I}$ to be smooth in the sense that the union of the images of these $SU(2)$-structures on the slices of $M^5 \times I$,

$$\hat{\mathcal{Q}} := \bigsqcup_{r\in I} \hat{\mathcal{Q}}_r \subset F\big(T(M^5 \times I)\big), \tag{5.1.3}$$

is an $SU(2)$-structure on $M^5 \times I$. $SU(2)$ is embedded into $SO(5)$ as upper-left block matrices, which in turn is embedded into $SO(6)$ as upper-left block matrices, such

that $SU(2)$ acts trivially on e_5 and e_6.
Let e_r be a smooth family of local frames on M^5, where e_r is adapted to $\mathcal{Q}_r$. This induces a local section of $\hat{\mathcal{Q}}$ on $M^5 \times I$ which we denote by e. Thus, on $M^5 \times I$ we are given globally well-defined sections having components with respect to e as follows:

$$\begin{aligned} \omega^\alpha &= \frac{1}{2}\, \eta^\alpha{}_{ab}\, \beta^a_r \wedge \beta^b_r \in \Omega^2(M^5 \times I), \\ \eta &= -\beta^5_r \in \Omega^1(M^5 \times I), \\ \mathrm{d}r &= \beta^6 \in \Omega^1(M^5 \times I). \end{aligned} \tag{5.1.4}$$

The foundation of the constructions in this chapter is the following proposition that provides a procedure to construct $SU(3)$-structures alternative to the one in [24].

Proposition 5.1.1: *Let M^5 be a 5-manifold and $\{\mathcal{Q}_r\}_{r\in I}$ be a one-parameter family of $SU(2)$-structures on M, which is smooth in the above sense.*

(1) If $(\eta_r,\, \omega^\alpha_r)_{r\in I}$ are the defining sections of $\mathcal{Q}_r$ and $\phi : I \to \mathbb{R}_+$, $r \mapsto \phi(r)$ is a smooth, positive function, then

$$\big(\tilde{\eta}_r,\, \tilde{\omega}^\alpha_r\big) = \big(\phi(r)\, \eta_r,\, \phi(r)^2\, \omega^\alpha_r\big)_{r\in I} \tag{5.1.5}$$

define a smooth family $\{\mathcal{Q}'_r\}_{r\in I}$ of $SU(2)$-structures on M^5.

(2) The pushforward $\widehat{\mathcal{Q}'}$ is an $SU(2)$-structure on $M^5 \times I$ that is compatible with the metric

$$\tilde{g} = \phi(r)^2\, g_r + \mathrm{d}r^2 \tag{5.1.6}$$

if g_r is the metric that $\mathcal{Q}_r$ is compatible with on M^5.

(3) $\widehat{\mathcal{Q}'}$ is contained in an $SU(3)$-structure P, which is given in terms of the defining sections

$$\omega = \tilde{\omega}^3 + \mathrm{d}r \wedge \tilde{\eta} = \phi^2\, \omega^3_r + \phi\, \mathrm{d}r \wedge \eta_r, \tag{5.1.7}$$

$$\Omega^+ = \tilde{\omega}^2 \wedge \tilde{\eta} - \tilde{\omega}^1 \wedge \mathrm{d}r = \phi^2\big(\phi\, \omega^2_r \wedge \eta_r - \omega^1_r \wedge \mathrm{d}r\big), \tag{5.1.8}$$

$$\Omega^- = -\big(\tilde{\omega}^1 \wedge \tilde{\eta} + \tilde{\omega}^2 \wedge \mathrm{d}r\big) = -\phi^2\big(\phi\, \omega^1_r \wedge \eta_r + \omega^2_r \wedge \mathrm{d}r\big). \tag{5.1.9}$$

Proof. (1) is just a conformal rescaling treated already in section 4.3.
$\widehat{\mathcal{Q}'}$ is an $SU(2)$-structure on $M^5 \times I$ by the same reasoning as we gave for $\hat{\mathcal{Q}}$. Since $\tilde{g}$ takes its standard components with respect to the rescaled coframes, $\widehat{\mathcal{Q}'}$ is compatible with $\tilde{g}$. Hence, (2) holds true as well.
We prove (3) by showing that $(\omega, \Omega^+, \Omega^-)$ have the standard components of the defining sections of an $SU(3)$-structure on a 6-manifold as in equations (3.1.14) and (3.1.22). If $\tilde{\beta}$ is a coframe adapted to $\widehat{\mathcal{Q}'}$, and with $\tilde{\beta}^6 = \beta^6 = \mathrm{d}r$, we have

$$\omega = \tilde{\omega}^3 + \mathrm{d}r \wedge \tilde{\eta} = \tilde{\beta}^1 \wedge \tilde{\beta}^2 + \tilde{\beta}^3 \wedge \tilde{\beta}^4 + \tilde{\beta}^5 \wedge \tilde{\beta}^6, \tag{5.1.10}$$

$$\begin{aligned}\Omega^+ &= \tilde{\omega}^2 \wedge \tilde{\eta} - \tilde{\omega}^1 \wedge \mathrm{d}r \\ &= \big(-\tilde{\beta}^1 \wedge \tilde{\beta}^3 + \tilde{\beta}^2 \wedge \tilde{\beta}^4 \big) \wedge (-\tilde{\beta}^5) - \big(\tilde{\beta}^1 \wedge \tilde{\beta}^4 + \tilde{\beta}^2 \wedge \tilde{\beta}^3 \big) \wedge \tilde{\beta}^6 \\ &= \tilde{\beta}^{135} - \tilde{\beta}^{245} - \tilde{\beta}^{146} - \tilde{\beta}^{236},\end{aligned} \tag{5.1.11}$$

$$\begin{aligned}\Omega^- &= -\big(\tilde{\omega}^1 \wedge \tilde{\eta} + \tilde{\omega}^2 \wedge \mathrm{d}r \big) \\ &= -\big(\tilde{\beta}^1 \wedge \tilde{\beta}^4 + \tilde{\beta}^2 \wedge \tilde{\beta}^3 \big) \wedge (-\tilde{\beta}^5) - \big(-\tilde{\beta}^1 \wedge \tilde{\beta}^3 + \tilde{\beta}^2 \wedge \tilde{\beta}^4 \big) \wedge \tilde{\beta}^6 \\ &= \tilde{\beta}^{145} + \tilde{\beta}^{235} + \tilde{\beta}^{136} - \tilde{\beta}^{246}.\end{aligned} \tag{5.1.12}$$

This is precisely the form that the defining sections of an $SU(3)$-structure on a 6-manifold take in a coframe compatible with the $SU(3)$-structure. Hence, this proves (3). □

Note that the transformation $(\eta, \omega^\alpha) \mapsto (\phi\,\eta, \phi^2\,\omega^\alpha)$ can be seen as induced by a transformation as introduced in definition 4.3.2. The above proposition yields a way to obtain 6-dimensional manifolds with an $SU(3)$-structure from 5-manifolds carrying families of $SU(2)$-structures.

It would, thus, be desirable to find a way of constructing families of $SU(2)$-structures on 5-manifolds. The first example that comes to mind is the case of a constant family, i. e. a single $SU(2)$-structure. This can always be lifted to an $SU(2)$-structure on $M^5 \times I$ and then extended to an ambient $SU(3)$-structure by the above proposition. Furthermore, we can construct smooth families of $SU(2)$-structures on 5-manifolds by considering a fixed $SU(2)$-structure and applying a smooth family of normal deformations as introduced in section 4.3. This yields a smooth one-parameter family of $SU(2)$-structures on M^5 that can then be lifted to $M^5 \times I$ via proposition 5.1.1. Such families of deformations are induced by families of maps $(h_r)_{r \in I}$ of the form

$$h_r : M^5 \mapsto N_{GL(5,\mathbb{R})}(SU(2)),\ x \mapsto h_r(x) \quad \forall\, r \in I. \tag{5.1.13}$$

In order for the resulting family $\mathcal{Q}_r = R_{h_r}\,\mathcal{Q}$ of $SU(2)$-structures on M^5 to be smooth, we need that

$$h : M^5 \times I \to N_{GL(5,\mathbb{R})}(SU(2)),\ (x, r) \mapsto h_r(x) \tag{5.1.14}$$

is smooth. We can, thus, view h either as a family of maps $(h_r)_{r \in I}$ defined on M^5, or as a smooth map h from $M^5 \times I$ to $N_{GL(5,\mathbb{R})}(SU(2)) \hookrightarrow N_{GL(6,\mathbb{R})}(SU(2))$. The embedding is induced by

$$GL(5,\mathbb{R}) \hookrightarrow GL(6,\mathbb{R}),\ L \mapsto \begin{bmatrix} L & \\ & 1 \end{bmatrix}. \tag{5.1.15}$$

We directly conclude:

Lemma 5.1.2: *Consider an $SU(2)$-structure $\mathcal{Q}$ on M^5 together with a smooth map $h : M^5{\times}I \to N_{GL(6,\mathbb{R})}(SU(2))$. Then there are two ways of constructing an $SU(2)$-structure on $M^5{\times}I$ from this data.*
First, one can apply the normal deformations given by h_r to $\mathcal{Q}$, obtaining a one-parameter family $\mathcal{Q}_r$ of $SU(2)$-structures on M^5, which can then be lifted to an $SU(2)$-structure on $M^5{\times}I$ via proposition 5.1.1.
Second, one can lift the constant family of $SU(2)$-structures given by $\mathcal{Q}_r = \mathcal{Q}$ to $M^5{\times}I$, and afterwards apply the deformation given by $h : M^5{\times}I \to N_{GL(6,\mathbb{R})}(SU(2))$ to the resulting $SU(2)$-structure, thus again obtaining an $SU(2)$-structure on $M^5{\times}I$.
The $SU(2)$-structures on $M^5{\times}I$ constructed by these means coincide. That is, applying a family of rotations and lifting via proposition 5.1.1 commute.

As $h = h(r)$, the $SU(2)$-structures resulting from the above construction are compatible with a ϕ-cone metric on $M^5{\times}I$. The ϕ-cones obtained this way carry an $SU(2)$-structure $\hat{\mathcal{Q}}$ that is contained in an $SU(3)$-structure $\mathcal{P}$ by the last part of proposition 5.1.1. We, therefore, have the following embeddings of principal fiber bundles as subbundles:

$$\hat{\mathcal{Q}} \hookrightarrow \mathcal{P} \hookrightarrow SO(M^5{\times}I, g_\phi). \tag{5.1.16}$$

From this we infer that the instanton bundle of $\hat{\mathcal{Q}}$ is a vector subbundle of the instanton bundle of $\mathcal{P}$,

$$W(\hat{\mathcal{Q}}) \subset W(\mathcal{P}). \tag{5.1.17}$$

In the following we will most of the time drop the hat of the lifted $SU(2)$-structure. We now turn to employing this general procedure in explicit constructions that can also be found in [39] and [40].

5.2 Kähler-Torsion Sine-Cones

To begin with, let us consider a 5-manifold with a Sasaki-Einstein $SU(2)$-structure and $I = (0, \Lambda\,\pi)$, where $\Lambda \in \mathbb{R}_+$ is a positive constant. As argued in section 4.3, a conformal rescaling of the frames belonging to the $SU(2)$-structure yields another $SU(2)$-structure. Thus, let us consider a family of conformal rescalings as induced by

$$h : M^5{\times}I \to N_{GL(6,\mathbb{R})}(SU(2)),\ h(x,r) = h_r(x) = \frac{1}{\Lambda\,\sin(\frac{r}{\Lambda})}\begin{bmatrix} \mathbb{1}_5 & \\ & 0 \end{bmatrix}, \tag{5.2.1}$$

i.e. $\beta^\mu \mapsto \Lambda\,\sin(\frac{r}{\Lambda})\,\beta^\mu,\ \beta^6 \mapsto \beta^6$. The resulting $SU(2)$-structure is an $SU(2)$-structure on the sine-cone

$$\left(M^5{\times}(0,\Lambda\,\pi),\, g = \Lambda^2\,\sin\left(\frac{r}{\Lambda}\right)^2 g_5 + \mathrm{d}r^2\right) \tag{5.2.2}$$

over M^5. We denote the unaltered pullbacks of η and the ω^α to the direct product $M^5\times I$ along the projection $\pi : M^5\times I \to M^5$, $(x,r) \mapsto x$ still by η and ω^α. These are the defining sections of the $SU(2)$-structure on $M^5\times I$ which results from lifting the constant family consisting of just the Sasaki-Einstein $SU(2)$-structure on M^5. We will often call this lift the *pushforward* $SU(2)$-structure on $M^5\times I$.
With this notation, the defining sections for the ambient $SU(3)$-structure for the deformed $SU(2)$-structure are given by

$$\begin{aligned} \omega &= \Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \omega^3 + \Lambda \sin\left(\frac{r}{\Lambda}\right) \mathrm{d}r \wedge \eta, \\ \Omega^+ &= \Lambda^3 \sin\left(\frac{r}{\Lambda}\right)^3 \omega^2 \wedge \eta - \Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \omega^1 \wedge \mathrm{d}r, \\ \Omega^- &= -\Lambda^3 \sin\left(\frac{r}{\Lambda}\right)^3 \omega^1 \wedge \eta - \Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \omega^2 \wedge \mathrm{d}r, \end{aligned} \tag{5.2.3}$$

according to (5.1.7). In order to classify this particular $SU(3)$-manifold, we compute its torsion classes.

Proposition 5.2.1: *The torsion classes of* $(M^5\times(0,\Lambda\pi),\, g)$ *endowed with the above* $SU(3)$*-structure read*

$$W_1 = W_2 = W_3 = 0, \quad W_4 = -\frac{2}{\Lambda} \tan\left(\frac{r}{2\Lambda}\right) \mathrm{d}r, \quad W_5 = \frac{3}{\Lambda} \tan\left(\frac{r}{2\Lambda}\right) \mathrm{d}r. \tag{5.2.4}$$

Proof. We compute

$$\mathrm{d}\omega = 2\Lambda \sin\left(\frac{r}{\Lambda}\right)\left(\cos\left(\frac{r}{\Lambda}\right) - 1\right) \mathrm{d}r \wedge \omega^3 = -\frac{2}{\Lambda} \tan\left(\frac{r}{2\Lambda}\right) \mathrm{d}r \wedge \tilde{\omega}^3, \tag{5.2.5}$$

$$\mathrm{d}\Omega^+ = 3\Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \omega^2 \wedge \mathrm{d}r \wedge \eta = -\frac{3}{\Lambda} \tan\left(\frac{r}{2\Lambda}\right) \tilde{\omega}^2 \wedge \mathrm{d}r \wedge \tilde{\eta}, \tag{5.2.6}$$

$$\mathrm{d}\Omega^- = -3\Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \omega^1 \wedge \mathrm{d}r \wedge \eta = -\frac{3}{\Lambda} \tan\left(\frac{r}{2\Lambda}\right) \tilde{\omega}^1 \wedge \mathrm{d}r \wedge \tilde{\eta}. \tag{5.2.7}$$

Since $\Omega \wedge \omega = 0$ and $W_3 \in \Omega^{2,1}(M) \oplus \Omega^{1,2}(M)$,

$$\mathrm{d}\omega \wedge \Omega^\pm = -W_1^\pm\, \omega \wedge \omega \wedge \omega = 0, \tag{5.2.8}$$

whence we obtain

$$W_1^\pm = 0. \tag{5.2.9}$$

Further,

$$W_4 = \frac{1}{2}\, \omega \lrcorner \mathrm{d}\omega \quad \text{and} \quad W_5 = \Omega^+ \lrcorner \mathrm{d}\Omega^+. \tag{5.2.10}$$

Note that the interior product $\lrcorner$ is taken with respect to the sine-cone metric in all these cases. Here this gives

$$W_4 = \frac{1}{2}\left(\tilde{\omega}^3 + \mathrm{d}r \wedge \tilde{\eta}\right) \lrcorner \left(-\frac{2}{\Lambda} \tan\left(\frac{r}{2\Lambda}\right) \mathrm{d}r \wedge \tilde{\omega}^3\right)$$

$$= -\frac{2}{\Lambda}\tan\left(\frac{r}{2\Lambda}\right)\mathrm{d}r \tag{5.2.11}$$

and

$$W_5 = \frac{1}{2}\left(\tilde{\omega^2}\wedge\tilde{\eta} - \tilde{\omega^1}\wedge \mathrm{d}r\right) \lrcorner \left(-\frac{3}{\Lambda}\tan\left(\frac{r}{2\Lambda}\right)\tilde{\omega}^2\wedge\mathrm{d}r\wedge\tilde{\eta}\right)$$
$$= \frac{3}{\Lambda}\tan\left(\frac{r}{2\Lambda}\right)\mathrm{d}r. \tag{5.2.12}$$

Now one can check that $\mathrm{d}\omega = W_4\wedge\omega$, implying $W_3 = 0$. □

The fact that $3\,W_4 + 2\,W_5 = 0$, with both W_4 and W_5 being exact and real, indicates that this $SU(3)$-structure is mapped to a Calabi-Yau $SU(3)$-structure via a conformal equivalence [22]. In fact, the conformal equivalence of the sine-cone to the cylinder over M^5 has been worked out explicitly for example in [39, 40]. On the other hand, the cone over M^5 has been shown to be Calabi-Yau, as well as conformally equivalent to the cylinder over M^5 e. g. in [2]. Composing these two conformal equivalences yields a conformal equivalence of the sine-cone under consideration here to the Calabi-Yau cone over M^5. This also maps the respective $SU(3)$-structures onto one another in consistency with the results obtained above.
Recall from section 3.1 that we defined an $SU(3)$-structure to be Kähler-torsion if its ambient $U(3)$-structure defined by (g,ω) is Kähler-torsion.

Corollary 5.2.2: *The $SU(3)$-structure (g,ω,Ω) on the sine-cone over a Sasaki-Einstein 5-manifold as considered here is Kähler-torsion.*

Proof. From the torsion classes (5.2.4) one directly sees that (M,g,ω) is a complex manifold. Regarding the Kähler-torsion property we make use of theorem 10.1 in [28], which states that the torsion 3-form of the canonical, or Bismut, connection of an almost Hermitian manifold is given by

$$\flat_g(T^B) = -J(\mathrm{d}\omega) + \flat_g(N). \tag{5.2.13}$$

Since $C_{\sin}(M)$ is complex, N vanishes identically and we have

$$\begin{aligned}\flat_g(T^B) &= -J(\mathrm{d}\omega) = -\frac{2}{\Lambda}\tilde{\eta}\wedge\tilde{\omega}^3\\ &= \frac{i}{2\Lambda}\tan\left(\frac{r}{2\Lambda}\right)(\theta^3+\theta^{\bar{3}})\wedge(\theta^1\wedge\theta^{\bar{1}}+\theta^2\wedge\theta^{\bar{2}}),\end{aligned} \tag{5.2.14}$$

where we made use of the θ^j introduced in (3.1.16). This is the real part of a $(2,1)$-form, whence the $SU(3)$-structure under consideration is indeed Kähler-torsion. □

However, $2\,W_4 + W_5 \neq 0$, such that this $SU(3)$-structure is not Calabi-Yau-torsion. In particular, the Bismut connection preserves g and ω, but fails to preserve Ω.

Note, furthermore, that in the limit $\Lambda \to \infty$ this Kähler-torsion $SU(3)$ sine-cone becomes precisely the Calabi-Yau metric cone over M mentioned above. In this limit, r tends to the standard coordinate on this metric cone, and, as $W_4, W_5 \to 0$, the Bismut connection converges to the Levi-Civita connection on the cone.

5.3 Nearly Kähler Sine-Cones

In this section we continue where we left off in the last section, namely at the Kähler-torsion $SU(3)$-structure on the sine-cone over a Sasaki-Einstein $SU(2)$ 5-manifold M^5. We construct a new $SU(3)$-structure on the sine-cone from this first one by applying a rotation in the sense of section 4.3.
With $\hat{\mu}, \hat{\nu} = 1, \ldots, 6$, we extend the 't Hooft symbol η^2 to a (6×6)-matrix by putting ${\eta^2}_{\hat{\mu}\hat{\nu}} = 0$ whenever $\hat{\mu} = 5, 6$ or $\hat{\nu} = 5, 6$. Consider the map

$$\mathcal{T} : M^5 \times (0, \Lambda\,\pi) \to SO(6),\ \mathcal{T}(x, r) = \exp\left(\frac{r}{2\Lambda}\,\eta^2\right), \tag{5.3.1}$$

or explicitly,

$$\mathcal{T}(x, r) = \begin{bmatrix} \cos(\frac{r}{2\Lambda}) & 0 & -\sin(\frac{r}{2\Lambda}) & 0 & \\ 0 & \cos(\frac{r}{2\Lambda}) & 0 & \sin(\frac{r}{2\Lambda}) & \\ \sin(\frac{r}{2\Lambda}) & 0 & \cos(\frac{r}{2\Lambda}) & 0 & \\ 0 & -\sin(\frac{r}{2\Lambda}) & 0 & \cos(\frac{r}{2\Lambda}) & \\ & & & & \mathbb{1}_2 \end{bmatrix}. \tag{5.3.2}$$

This can be understood as a family $(\mathcal{T}_r)_{r\in(0,\Lambda\,\pi)}$ of maps from M to $SO(5)$, which is embedded into $SO(6)$ as upper-left block matrices, as elaborated on in section 5.1. The rotation induced by $\mathcal{T}$ is applied to the $SU(2)$-structure $\mathcal{Q}$ on the sine-cone, such that the induced transformation of the coframes of $\mathcal{Q}$ given by

$$\beta^{\hat{\mu}} \mapsto {\mathcal{T}^{\hat{\mu}}}_{\hat{\nu}}\ \beta^{\hat{\nu}}. \tag{5.3.3}$$

This corresponds to acting on $F(T(M^5 \times I))$ with $R_{\mathcal{T}^{-1}}$. We may first consider the family $(\mathcal{T}_r)_{r\in(0,\Lambda\,\pi)}$ of rotations on M in 5 dimensions.

Lemma 5.3.1: *Let M^5 be a 5-manifold endowed with a Sasaki-Einstein $SU(2)$-structure $\mathcal{Q}_5$ defined by (η, ω^α) and be $(\mathcal{T}_r)_{r\in(0,\Lambda\,\pi)}$ as defined above.*
Then $R_{\mathcal{T}_r^{-1}}\,\mathcal{Q}_5 =: \mathcal{Q}'_{5,r}$ is a different $SU(2)$-structure on M defined by

$$\begin{aligned} \eta_r &= \eta, \\ \omega_r^1 &= \cos\left(\frac{r}{\Lambda}\right)\omega^1 - \sin\left(\frac{r}{\Lambda}\right)\omega^3, \\ \omega_r^2 &= \omega^2, \\ \omega_r^3 &= \cos\left(\frac{r}{\Lambda}\right)\omega^3 + \sin\left(\frac{r}{\Lambda}\right)\omega^1. \end{aligned} \tag{5.3.4}$$

Proof. We only have to show that $(\eta_r, \omega_r^\alpha)$ take the standard components (3.2.4) with respect to the rotated bases

$$\beta_r^\mu := (\mathcal{T}_r)^\mu{}_\nu\, \beta^\nu. \tag{5.3.5}$$

Making use of the shorthand notation $\beta^{a_1 \cdots a_k} := \beta^{a_1} \wedge \ldots \wedge \beta^{a_k}$ we compute

$$\begin{aligned} \omega_r^1 &= \cos\left(\frac{r}{\Lambda}\right)\omega^1 - \sin\left(\frac{r}{\Lambda}\right)\omega^3 \\ &= \left(\cos\left(\frac{r}{2\Lambda}\right)^2 - \sin\left(\frac{r}{2\Lambda}\right)^2\right)(\beta^{14} + \beta^{23}) \\ &\qquad - 2\cos\left(\frac{r}{2\Lambda}\right)\sin\left(\frac{r}{2\Lambda}\right)(\beta^{12} + \beta^{34}) \\ &= \beta_r^{14} + \beta_r^{23}. \end{aligned} \tag{5.3.6}$$

Furthermore,

$$\begin{aligned} \omega_r^2 &= \omega^2 \\ &= -\left(\cos\left(\frac{r}{2\Lambda}\right)^2 + \sin\left(\frac{r}{2\Lambda}\right)^2\right)(\beta^{13} + \beta^{24}) \\ &= -\beta_r^{13} + \beta_r^{24} \end{aligned} \tag{5.3.7}$$

and

$$\begin{aligned} \omega_r^3 &= \cos\left(\frac{r}{\Lambda}\right)\omega^3 + \sin\left(\frac{r}{\Lambda}\right)\omega^1 \\ &= \left(\cos\left(\frac{r}{2\Lambda}\right)^2 - \sin\left(\frac{r}{2\Lambda}\right)^2\right)(\beta^{12} + \beta^{34}) \\ &\qquad + 2\cos\left(\frac{r}{2\Lambda}\right)\sin\left(\frac{r}{2\Lambda}\right)(\beta^{14} + \beta^{23}) \\ &= \beta_r^{12} + \beta_r^{34}. \end{aligned} \tag{5.3.8}$$

The computation for η is trivial, since β^5 is invariant under all $\mathcal{T}_r$, $r \in (0, \Lambda\,\pi)$. □

We write $(\tilde\eta, \tilde\omega^\alpha)$ for the lifts of the $(\eta,_r\, \omega_r^\alpha)$ to the sine-cone, and by proposition 5.1.1, we obtain another set of defining sections for an $SU(3)$-structure on the sine-cone. It is given by (g, ω, Ω), where g is the sine-cone metric and ω and Ω are as in (5.1.7).

Proposition 5.3.2: *The $SU(3)$-structure on the sine-cone over a Sasaki-Einstein 5-manifold as constructed here is nearly Kähler.*

Proof. First, we compute

$$\mathrm{d}\tilde\eta = \mathrm{d}\Big(\Lambda \sin\left(\frac{r}{\Lambda}\right)\eta\Big) = \cos\left(\frac{r}{\Lambda}\right)\mathrm{d}r \wedge \eta + 2\Lambda\,\sin\left(\frac{r}{\Lambda}\right)\omega^3, \tag{5.3.9}$$

$$\begin{aligned} \mathrm{d}\tilde{\omega}^1 &= \mathrm{d}\Big(\Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \Big(\cos\left(\frac{r}{\Lambda}\right)\omega^1 - \sin\left(\frac{r}{\Lambda}\right)\omega^3\Big)\Big) \\ &= -\frac{3}{\Lambda}\,\mathrm{d}r \wedge \tilde{\omega}^3 + 2\Lambda \sin\left(\frac{r}{\Lambda}\right)\mathrm{d}r \wedge \omega^1 - 3\Lambda \sin\left(\frac{r}{\Lambda}\right)\cos\left(\frac{r}{\Lambda}\right)\tilde{\eta} \wedge \omega^2, \end{aligned} \tag{5.3.10}$$

$$\mathrm{d}\tilde{\omega}^2 = \mathrm{d}\Big(\Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \omega^2\Big) = \frac{2}{\Lambda}\cos\left(\frac{r}{\Lambda}\right)\mathrm{d}r \wedge \tilde{\omega}^2 + 3\Lambda \sin\left(\frac{r}{\Lambda}\right)\tilde{\eta} \wedge \omega^1, \tag{5.3.11}$$

$$\begin{aligned} \mathrm{d}\tilde{\omega}^3 &= \mathrm{d}\Big(\Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \Big(\cos\left(\frac{r}{\Lambda}\right)\omega^3 + \sin\left(\frac{r}{\Lambda}\right)\omega^1\Big)\Big) \\ &= \frac{3}{\Lambda}\,\mathrm{d}r \wedge \tilde{\omega}^1 + 2\Lambda \sin\left(\frac{r}{\Lambda}\right)\mathrm{d}r \wedge \omega^3 - \frac{3}{\Lambda}\,\tilde{\eta} \wedge \tilde{\omega}^2. \end{aligned} \tag{5.3.12}$$

Hence, we have

$$\mathrm{d}\omega = \mathrm{d}\Big(\tilde{\omega}^3 + \mathrm{d}r \wedge \tilde{\eta}\Big) = \frac{3}{\Lambda}\,\mathrm{d}r \wedge \tilde{\omega}^1 - \frac{3}{\Lambda}\,\tilde{\eta} \wedge \tilde{\omega}^2 = -\frac{3}{\Lambda}\,\Omega^+. \tag{5.3.13}$$

On the other hand,

$$\begin{aligned} \mathrm{d}\Omega^- &= -\,\mathrm{d}\Big(\tilde{\omega}^1 \wedge \tilde{\eta} + \tilde{\omega}^2 \wedge \mathrm{d}r\Big) \\ &= \frac{2}{\Lambda}\,\tilde{\omega}^3 \wedge \tilde{\omega}^3 + \frac{4}{\Lambda}\,\tilde{\omega}^3 \wedge \mathrm{d}r \wedge \tilde{\eta} \\ &= \frac{2}{\Lambda}\,\omega \wedge \omega, \end{aligned} \tag{5.3.14}$$

whence this $SU(3)$-structure is nearly Kähler, by definition 3.1.5. Its only non-vanishing torsion class is $W_1^- = \lambda = \frac{2}{\Lambda}$. □

By writing ω and Ω in terms of η and the ω^α, one can see that we have constructed the same nearly Kähler sine-cone over Sasaki-Einstein 5-manifolds which has been obtained in [24] by the use of flow equations. We, in contrast, gave a more explicit way of arriving at this nearly Kähler sine-cone by means of deformations as discussed in section 4.3 and proposition 5.1.1.
In the limit $\Lambda \to \infty$, the rotation $\mathcal{T}$ becomes trivial and the metric approaches the cone metric. In this large-volume limit, the nearly Kähler $SU(3)$-structure sine-cone tends to the Calabi-Yau cone since $W_1 \to 0$.

5.4 Half-Flat Cylinders

The third construction we would like to consider is an $SU(3)$-structure on the direct product $M^5 \times I$. The interval can very well be the complete real line. In this case the resulting Riemannian manifold $M^5 \times I$ endowed with the direct-product metric is usually called the cylinder over M^5. We will, however, even for a bounded interval I call $M^5 \times I$ with the direct-product metric a cylinder over M^5.

We lift the Sasaki-Einstein $SU(2)$-structure of M^5 to the direct product and employ the deformation

$$\begin{aligned} \beta_z^1 &= \cos(\zeta)\,\beta^4 + \sin(\zeta)\,\beta^3, & \beta_z^2 &= -\beta^1, \\ \beta_z^3 &= \beta^2, & \beta_z^4 &= \cos(\zeta)\,\beta^3 - \sin(\zeta)\,\beta^4, \\ \beta_z^5 &= \varrho\,\beta^5, & \beta_z^6 &= \mathrm{d}r = \beta^6, \end{aligned} \tag{5.4.1}$$

where $\zeta \in [0, 2\pi]$ and $\varrho \in \mathbb{R}_+$ are two constant parameters. The resulting Riemannian manifold is the cylinder over the Riemannian manifold

$$(M^5,\, g_{5,\varrho} = \delta_{ab}\,\beta^a \otimes \beta^b + \varrho^2\,\beta^5 \otimes \beta^5). \tag{5.4.2}$$

Lemma 5.4.1: *Let M^5 be a 5-manifold carrying a Sasaki-Einstein $SU(2)$-structure $\mathcal{Q}_5$ defined by (η, ω^α), and let β_z be as defined above.*
Then these new coframes define a new $SU(2)$-structure $\mathcal{Q}_{5,(\zeta,\varrho)}$ on M^5 for all parameter values $(\zeta, \varrho) \in [0, 2\pi] \times \mathbb{R}_+$, which is defined by

$$\begin{aligned} \eta_z &= \varrho\,\eta, \\ \omega_z^1 &= -\,\omega^3, \\ \omega_z^2 &= \cos(\zeta)\,\omega^2 + \sin(\zeta)\,\omega^1, \\ \omega_z^3 &= \cos(\zeta)\,\omega^1 - \sin(\zeta)\,\omega^2. \end{aligned} \tag{5.4.3}$$

Proof. A direct computation shows

$$\begin{aligned} \omega_z^1 &= -\big(\cos(\zeta)^2 + \sin(\zeta)^2\big)\,\beta^{12} - \beta^{34} = \beta_z^{14} + \beta_z^{23}, \\ \omega_z^2 &= \cos(\zeta)\,\big(-\beta^{13} + \beta^{24}\big) + \sin(\zeta)\,\big(\beta^{14} + \beta^{23}\big) = -\beta_z^{13} + \beta_z^{24}, \\ \omega_z^3 &= \cos(\zeta)\,\big(\beta^{14} + \beta^{23}\big) + \sin(\zeta)\,\big(\beta^{13} - \beta^{24}\big) = \beta_z^{12} + \beta_z^{34}. \end{aligned} \tag{5.4.4}$$

This already completes the proof, since with respect to the coframes β_z the defining forms $(\eta_z, \omega_z^\alpha)$ have the standard components of defining sections of an $SU(2)$-structure. □

Note that we do not consider the parameters ζ and ϱ to be related to the cone direction, in contrast to the nearly Kähler construction. In the present case, ζ and ϱ are constant, free parameters. Employing the rotation on M^5 for fixed values of the parameters, lifting the resulting $SU(2)$-structure to $M^5 \times I$ and extending it to an $SU(3)$-structure defined by (ω, Ω) from $(\eta_z, \omega_z^\alpha)$ as in (5.1.7), we arrive at

Proposition 5.4.2: *The $SU(3)$-structure on the cylinder over $(M^5,\, g_{5,\varrho})$ defined* *by* (ω, Ω) *has the following torsion classes:*

$$\begin{aligned} W_1^- &= \frac{3+2\varrho^2}{3\varrho}, & W_1^+ &= 0, \\ W_2^- &= \frac{4\varrho^2-3}{3\varrho}\,\big(\omega_z^3 - 2\,\mathrm{d}r \wedge \eta_z\big), & W_2^+ &= 0, \\ W_3 &= \frac{2\varrho^2-3}{2\varrho}\,\big(\omega_z^2 \wedge \mathrm{d}r + \omega_z^1 \wedge \eta_z\big), & W_4 &= 0, \quad W_5 = 0. \end{aligned} \tag{5.4.5}$$

Proof. First, we compute the differentials

$$\mathrm{d}\omega = \mathrm{d}\big(\omega_z^3 + \mathrm{d}r \wedge \eta_z\big) = -\frac{3}{\varrho}\,\omega_z^2 \wedge \eta_z + 2\varrho\,\omega_z^1 \wedge \mathrm{d}r, \tag{5.4.6}$$

$$\mathrm{d}\Omega^+ = \mathrm{d}\big(\omega_z^2 \wedge \eta_z - \omega_z^1 \wedge \mathrm{d}r\big) = 0, \tag{5.4.7}$$

$$\mathrm{d}\Omega^- = -\,\mathrm{d}\big(\omega_z^1 \wedge \eta_z + \omega_z^2 \wedge \mathrm{d}r\big) = \frac{3}{\varrho}\,\omega_z^3 \wedge \mathrm{d}r \wedge \eta_z + 2\varrho\,\omega_z^3 \wedge \omega_z^3. \tag{5.4.8}$$

Then we compare

$$\mathrm{d}\Omega^- \wedge \omega = \Big(\frac{3}{\varrho} + 2\Big)\omega_z^3 \wedge \omega_z^3 \wedge \mathrm{d}r \wedge \eta_z \tag{5.4.9}$$

to

$$\omega \wedge \omega \wedge \omega = 3\,\omega_z^3 \wedge \omega_z^3 \wedge \mathrm{d}r \wedge \eta_z. \tag{5.4.10}$$

Thereby, we obtain

$$W_1^- = \frac{3+2\varrho^2}{3\varrho} \quad \text{and} \quad W_1^+ = 0. \tag{5.4.11}$$

We proceed with

$$W_4 = \frac{1}{2}\,\omega \lrcorner \mathrm{d}\omega = 0 \quad \text{and} \quad W_5 = \Omega^+ \lrcorner \mathrm{d}\Omega^+ = 0. \tag{5.4.12}$$

This leaves us with

$$\mathrm{d}\omega = -\frac{3}{2}\,W_1^-\,\Omega^+ + W_3, \tag{5.4.13}$$

that is,

$$\begin{aligned} W_3 &= -\,\frac{3}{\varrho}\,\omega_z^2 \wedge \eta_z + 2\varrho\,\omega_z^1 \wedge \mathrm{d}r + \frac{3+2\varrho^2}{2\varrho}\,\big(\omega_z^2 \wedge \mathrm{d}r - \omega_z^1 \wedge \eta_z\big) \\ &= \frac{2\varrho^2-3}{2\varrho}\,\big(\omega_z^1 \wedge \mathrm{d}r + \omega_z^2 \wedge \eta_z\big). \end{aligned} \tag{5.4.14}$$

Finally, there is

$$\mathrm{d}\Omega = i\,W_1^-\,\omega \wedge \omega + \big(W_2^+ + i\,W_2^-\big) \wedge \omega, \tag{5.4.15}$$

whence

$$\left(W_2^+ + i\,W_2^-\right) \wedge \omega = i\,\frac{4\varrho^2 - 3}{3\varrho}\left(\omega_z^3 \wedge \omega_z^3 - \omega_z^3 \wedge \mathrm{d}r \wedge \eta_z\right). \tag{5.4.16}$$

Thus, we arrive at

$$W_2^+ = 0 \quad \text{and} \quad W_2^- = \frac{3 - 4\varrho^2}{3\varrho}\left(\omega_z^3 - 2\,\mathrm{d}r \wedge \eta_z\right). \tag{5.4.17}$$

Note that $\omega \lrcorner W_2 = 0$ and $\omega \lrcorner W_3 = 0$ are indeed satisfied. □

By definition 3.1.5, this directly leads to

Corollary 5.4.3: *The $SU(3)$-structure on the cylinder over $(M^5,\, g_{5,\varrho})$ defined by (ω, Ω) is half-flat for all $(\zeta, \varrho) \in [0, 2\pi] \times \mathbb{R}_+$.*

We have constructed a two-parameter family of $SU(2)$-structures starting from a Sasaki-Einstein $SU(2)$-structure in five dimensions and lifted this family to a two-parameter family of $SU(3)$-structures on $M^5 \times I$, where I is an open interval. All $SU(3)$-structures obtained this way are half-flat. Note that

$$W_2 = 0 \quad \Leftrightarrow \quad \varrho = \frac{\sqrt{3}}{2}. \tag{5.4.18}$$

This will be of interest in section 6.6.

5.5 Relevance for String Compactifications

In this chapter we constructed 6-dimensional manifolds with an $SU(3)$-structure from 5-manifolds endowed with a Sasaki-Einstein $SU(2)$-structure. The Kähler-torsion and nearly Kähler sine-cones obtained this way are 6-manifolds which can be extended to compact 6-manifolds, possibly with conical singularities, by taking their topological closure. However, note that due to the sine-factors, the sections defining the $SU(3)$-structures tend to zero at the tips of the sine-cone. Thus, even in cases where the metric does not become singular at the tips, as for example for M^5 being the round 5-sphere, the $SU(3)$-structures we constructed here will not extend to the completion of the sine-cone. However, note that as (M^5, g_5) is Einstein for the sine-cone constructions, the sine-cone will be Einstein as well. Due to this fact, the sine-cones are of constant scalar curvature, whence there will be no singularities in at least the scalar curvature at the tips.
Both $SU(3)$-structures on the sine-cones can be seen as defined by a Killing spinor. In [12] it has been shown that sine-cones over Killing-spinor manifolds are again endowed with a Killing spinor. This is similar to Bär's famous theorem that the metric

cones over Killing spinor manifolds carry a parallel spinor [38]. For nearly Kähler spaces the Killing-spinor property is proven for example in [28, 41]. The sine-cones constructed in this chapter are, therefore, suitable to appear in flux compactifications of heterotic string theory.

Half-flat $SU(3)$-structures turn out to be too broad a class to be defined by Killing spinors. Instead, they are defined by generalized Killing spinors [30]. These are nowhere-vanishing sections of the spinor bundle of M^6, but in general the satisfy $\nabla^g_X \epsilon = \gamma(A(X))(\epsilon)$ with a non-trivial tensor field $A \in End(TM)$. As these structures are still defined by a spinor, one can still formulate the spinorial version of the instanton condition (4.2.5). However, this seems to no longer imply the (torsion free) Yang-Mills equations, such that the instanton condition might be an additional constraint rather than already implying the field equation for the gauge field in heterotic supergravity. Half-flat structures are prominent in heterotic flux compactifications, see e. g. [42].

Note that as long as we consider a bounded interval in section 5.4, the resulting half-flat manifolds extend to compact spaces, since all the structure can be extended to the boundary values of the cone parameter smoothly. These may, therefore, well be considered as internal spaces in flux compactifications of the heterotic string.

Chapter 6

Construction of Instantons

6.1 Instantons on Conical 6-Manifolds: General Idea

With the background on the geometry of instantons accumulated in chapter 4 at hand, we use knowledge about G-structures and defining sections elaborated in chapter 2 to formulate a general ansatz for the reduction of the instanton equation on conical manifolds, and use this to obtain explicit instanton solutions on the spaces constructed in chapter 5. The principal idea of the constructions in this sections has already appeared in [1]. Here, we formalize these considerations, such that some results and constraints become clearer. In particular, the origin of the constraint (6.1.12) becomes very clear in these geometrical terms.

In the following we assume (M, g) to be a D-dimensional Riemannian manifold endowed with a G-structure $\mathcal{Q}$, and we consider a gauge principal bundle $(\mathfrak{B}, \pi, M, H)$ over M as additional data. Suppose that, as principal bundles, $\mathcal{Q}$ is a bundle reduction of $\mathfrak{B}$. That is, we have the following structure of bundle reductions.

$$\begin{array}{ccc} & \mathcal{Q} \times G & \\ {\scriptstyle \iota_{\mathcal{Q},SO} \times \iota_G} \swarrow & & \searrow {\scriptstyle \Lambda \times \lambda} \\ SO(M,g) \times SO(D) & & \mathfrak{B} \times H \end{array} \tag{6.1.1}$$

In particular, there is an open covering $\{(U_\sigma, e_\sigma)\}_{\sigma \in \Sigma}$ of M by local sections of $\mathcal{Q}$. In the above situation, this is an open covering of M by local sections of $SO(M, g)$ as well, since $\mathcal{Q}$ is a subbundle of $SO(M, g)$.
Let $\rho : G \to GL(\mathbb{R}^D)$ the representation that associates TM to $\mathcal{Q}$, i. e.

$$TM = SO(M, g) \times_{(SO(D),\rho)} \mathbb{R}^D = \mathcal{Q} \times_{(G,\rho|_G)} \mathbb{R}^D. \tag{6.1.2}$$

Recall that if v_μ, $\mu = 1, \ldots, D$ is a basis of $\mathbb{R}^D$ and θ^μ, $\mu = 1, \ldots, D$ is a basis of $\mathbb{R}^{D*}$, we have

$$(e_\sigma)_\mu = [e_\sigma, v_\mu] \in \Gamma(T_U M) \quad \text{and} \quad (\beta_\sigma)^\mu = [e_\sigma, \theta^\mu] \in \Gamma(T^*_U M) \tag{6.1.3}$$

in the language of associated vector bundles. We may thus write

$$e_\sigma(x) = \big(e_{\sigma\, 1\,|x}, \ldots, e_{\sigma\, D\,|x}\big) \tag{6.1.4}$$

as a local orthonormal frame of $T_{U_\sigma}M$ for all $x \in U_\sigma$. Additionally, recall that if the representation on $\mathbb{R}^D$ reads $\rho(g)(v_\mu) = \rho(g)^\nu{}_\mu v_\nu$, the representation on $(\mathbb{R}^D)^*$ is ρ^{-T}, and

$$\rho^{-T}(g)(\theta^\mu) = \big(\rho^{-T}(g)\big)_\nu{}^\mu \theta^\nu = \big(\rho^T(g^{-1})\big)_\nu{}^\mu \theta^\nu = \rho(g^{-1})^\mu{}_\nu \theta^\nu. \tag{6.1.5}$$

Furthermore, since $\mathcal{Q}$ is a bundle reduction of $\mathfrak{B}$, the open covering $\{(U_\sigma, e_\sigma)\}_{\sigma\in\Sigma}$ induces an open covering $\{(U_\sigma, (\Lambda \circ e)_\sigma)\}_{\sigma\in\Sigma}$ of M by local sections of $\mathfrak{B}$. For compactness of notation we write $\hat{e}_\sigma \coloneqq \Lambda \circ e_\sigma$.
Now, let $A \in \mathcal{C}(\mathcal{Q})$ be an instanton for $\mathcal{Q}$. This induces a connection $\hat{A} \in \mathcal{C}(\mathfrak{B})$, which arises from the pushforward of the horizontal tangent spaces on $\mathcal{Q}$ via $\Lambda^* A$. In particular, A and $\hat{A}$ are related as (cf. [18] or appendix A.2),

$$\hat{e}_\sigma{}^* \hat{A} = (\Lambda \circ e_\sigma)^* \hat{A} = \lambda_* \circ \big(e_\sigma{}^* A\big) \;\forall\, \sigma \in \Sigma. \tag{6.1.6}$$

Our goal is to find other instantons that are connections on $\mathfrak{B}$. As $\mathcal{C}(\mathfrak{B})$ is an affine vector space over $\Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H,Ad_H)}$, we can write any $A' \in \mathcal{C}(\mathfrak{B})$ as

$$A' = \hat{A} + X, \quad \text{where } X \in \Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H,Ad_H)}. \tag{6.1.7}$$

In order to check whether such a connection is an instanton, we have to compute its field strength. To this end, let $\{I_B \,|\, B = 1, \ldots, dim(G)\}$ be a basis of $\mathfrak{g}$. Additionally, the local representations of X with respect to e_σ are given by

$$\big(\hat{e}_\sigma{}^* X\big)_\mu \coloneqq \big(\hat{e}_\sigma{}^* X\big)\big((e_\sigma)_\mu\big). \tag{6.1.8}$$

Lemma 6.1.1: *With respect to the orthonormal coframe $(\beta_\sigma)^\mu$ and with the notation introduced above, the local representation of the field strength of $A' = \hat{A} + X$ reads*

$$\begin{aligned} \hat{e}_\sigma{}^* F^{\hat{A}+X} &= \lambda_* \circ e_\sigma{}^* F^A + \mathrm{d}\big(\hat{e}_\sigma{}^* X\big)_\mu \wedge (\beta_\sigma)^\mu \\ &\quad + \frac{1}{2}\Big(T^\mu_{\ \nu\kappa}\big(\hat{e}_\sigma{}^* X\big)_\mu + \big[\big(\hat{e}_\sigma{}^* X\big)_\nu, \big(\hat{e}_\sigma{}^* X\big)_\kappa\big]\Big) \otimes (\beta_\sigma)^\nu \wedge (\beta_\sigma)^\kappa \\ &\quad + \Big(\big[\lambda_*(I_B), \big(\hat{e}_\sigma{}^* X\big)_\mu\big] - \big(\rho_*(I_B)\big)^\nu{}_\mu \big(\hat{e}_\sigma{}^* X\big)_\nu\Big) \otimes (e_\sigma{}^* A)^B \wedge (\beta_\sigma)^\mu. \end{aligned} \tag{6.1.9}$$

$T^\mu{}_{\nu\rho}$ is the torsion of A as a connection on $F(TM)$ with respect to the frame e_σ.

Proof. We start from the generic form of the field strength and compute

$$\begin{aligned} \hat{e}_\sigma{}^* F^{\hat{A}+X} &= \hat{e}_\sigma{}^*\Big(\mathrm{d}(\hat{A} + X) + \frac{1}{2}\, ad(\hat{A} + X) \overset{\circ}{\wedge} (\hat{A} + X)\Big) \\ &= \hat{e}_\sigma{}^* F^{\hat{A}} + \hat{e}_\sigma{}^* \mathrm{d}X + \hat{e}_\sigma{}^*\big(ad(\hat{A}) \overset{\circ}{\wedge} (X)\big) + \frac{1}{2}\,\hat{e}_\sigma{}^*\big(ad(X) \overset{\circ}{\wedge} (X)\big) \end{aligned}$$

$$
\begin{aligned}
&= \lambda_* \circ {e_\sigma}^* F^A + \mathrm{d}({\hat{e}_\sigma}^* X)_\mu \otimes (\beta_\sigma)^\mu + ({\hat{e}_\sigma}^* X)_\mu \otimes \mathrm{d}(\beta_\sigma)^\mu \\
&\quad + ad({\hat{e}_\sigma}^* \hat{A}) \overset{\circ}{\wedge} ({\hat{e}_\sigma}^* X) + \frac{1}{2} ad\big(({\hat{e}_\sigma}^* X)_\nu\big)\big(({\hat{e}_\sigma}^* X)_\kappa\big) \otimes (\beta_\sigma)^\nu \wedge (\beta_\sigma)^\kappa .
\end{aligned}
\tag{6.1.10}
$$

Here we have used that pullbacks and exterior differentials commute. Now we make use of ${\hat{e}_\sigma}^* \hat{A} = \lambda_* \circ {e_\sigma}^* A = ({s_\sigma}^* A)^B \otimes \lambda_*(I_B)$ and write $ad(\xi_1)(\xi_2) = [\xi_1, \xi_2]$. Employing the Maurer-Cartan identity we obtain

$$
\begin{aligned}
\mathrm{d}(\beta_\sigma)^\mu &= -\rho_*({e_\sigma}^* A) \overset{\circ}{\wedge} (\beta_\sigma)^\mu + T^\mu \\
&= -\big(\rho_*(I_B)\big)^\mu{}_\nu \, ({e_\sigma}^* A)^B \wedge (\beta_\sigma)^\mu + \frac{1}{2} T^\mu_{\nu\kappa} \, (\beta_\sigma)^\nu \wedge (\beta_\sigma)^\kappa ,
\end{aligned}
\tag{6.1.11}
$$

where $T \in \Omega^2(M, TM)$ is the torsion tensor of A as a connection on $F(TM)$. This directly yields (6.1.9). □

Thereby, we are provided with an expression for the field strength of A' in terms of the field strength and the torsion of A, and the local components $({\hat{e}_\sigma}^* X)_\nu$ of the Ad-equivariant, horizontal 1-form X on $\mathfrak{B}$.
While we are comfortable working with the quantities derived from A, we do not have much control over the components of X. In particular, in order to formulate an ansatz for a possible instanton, we would have to assign a value to these components for every $x \in M$. Even more, we would have to specify a certain covering $\{(U_\sigma, e_\sigma)\}_{\sigma \in \Sigma}$ of M, for the components of any section of associated bundles depend on the chosen trivialization, i. e. the chosen local frames. In other words, the local representations of X as used in the above lemma strongly depend on the choice of the local sections. However, on a generic manifold with a generic G-structure, there is no way of explicitly choosing such a covering of M by local frames.
Thus, the only way we have access to the components $({\hat{e}_\sigma}^* X)_\nu$ of X is to choose X in a way such that its components are independent of the choice of local sections of $\mathcal{Q}$. This turns out to be possible, but imposes certain severe constraints on X.

Lemma 6.1.2: *Consider a G-structure $\mathcal{Q}$ on M and a gauge bundle $(\mathfrak{B}, \pi, M, H)$ over M which are related by bundle reduction as above. Let ρ denote the representation that associates TM to $\mathcal{Q}$.*
Then, if G is connected, X has the same components with respect to any *two local sections $e, e' : U \to \mathcal{Q}$ if and only if with respect to either of the sections*

$$
\big[\lambda_*(I_B), \big({\hat{e}_\sigma}^* X\big)_\mu\big] - \big(\rho_*(I_B)\big)^\nu{}_\mu \, ({\hat{e}_\sigma}^* X)_\nu = 0. \tag{6.1.12}
$$

Proof. The local representation of X with respect to e is given by $\hat{e}^* X = (\Lambda \circ e)^* X$. We decompose this with respect to the local frame $e = (e_\mu)$. That is, as before,

$$
(\hat{e}^* X)_\mu = \hat{e}^* X \, (e_\mu). \tag{6.1.13}
$$

Now let us consider a different section e' that is related to e by means of the transition map $g : U \to G$, i. e.

$$e' = R_g \circ e. \tag{6.1.14}$$

Then we have

$$\hat{e}'^* X = (\Lambda \circ e')^* X = (R_{\lambda(g)} \circ \Lambda \circ e)^* X = Ad_H\big(\lambda(g^{-1})\big) \circ \hat{e}^* X \tag{6.1.15}$$

and

$$e'_\mu = [e', v_\nu] = [(R_g \circ e), v_\mu] = [e, \rho(g)(v_\mu)] = \rho(g)^\nu{}_\mu \, e_\nu. \tag{6.1.16}$$

Hence, we may compute

$$\begin{aligned}(\hat{e}'^* X)_\mu &= \hat{e}'^* X\,(e'_\mu) \\ &= \big(Ad_H\big(\lambda(g^{-1})\big) \circ \hat{e}^* X\big)\big(\rho(g)^\nu{}_\mu \, e_\nu\big) \\ &= Ad_H\big(\lambda(g^{-1})\big) \circ \rho(g)^\nu{}_\mu \,(\hat{e}^* X)_\nu.\end{aligned} \tag{6.1.17}$$

We observe that this equals $(\hat{e}^* X)_\nu$ for *any* choice of e and e' if and only if

$$(\hat{e}^* X)_\mu = Ad_H\big(\lambda(g^{-1})\big) \circ \rho(g)^\nu{}_\mu \,(\hat{e}^* X)_\nu \quad \forall\, g \in G. \tag{6.1.18}$$

This is merely the constraint which states that $(\hat{e}^* X)_{\mu|x}$ is an invariant element in the (anti-)representation $g \mapsto \rho^T(g) \otimes Ad(\lambda(g^{-1}))$. By putting $g = \exp(t\, I_B)$ and taking the derivative with respect to t at $t = 0$, this implies the infinitesimal version of this invariance:

$$0 = \big[\lambda_*(I_B), (\hat{e}^* X)_\nu\big] - \rho_*(I_B))^\mu{}_\nu \,(\hat{e}^* X)_\mu. \tag{6.1.19}$$

Conversely, by the exponential map, infinitesimal invariance under linear representations integrates to finite invariance for connected G. □

Thus, whenever (6.1.12) is satisfied for an $X \in \Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H, Ad_H)}$, it has *frame-independent* components with respect to local sections of $\mathcal{Q}$ and their images in $\mathfrak{B}$ under the bundle reduction Λ.

Assuming this property in (6.1.9), the field strength of $\hat{A} + X$ gets simplified.

Corollary 6.1.3: *In the above situation, if X has frame-independent components, the local representation of the field strength of $A' = \hat{A} + X$ reads*

$$\begin{aligned}\hat{e}_\sigma{}^* F^{\hat{A}+X} &= \lambda_* \circ e_\sigma{}^* F^A + \mathrm{d}\big(\hat{e}_\sigma{}^* X\big)_\mu \wedge (\beta_\sigma)^\mu \\ &+ \frac{1}{2}\Big(T^\mu_{\nu\kappa}\,\big(\hat{e}_\sigma{}^* X\big)_\mu + \big[\big(\hat{e}_\sigma{}^* X\big)_\nu, \big(\hat{e}_\sigma{}^* X\big)_\kappa\big]\Big) \otimes (\beta_\sigma)^\nu \wedge (\beta_\sigma)^\kappa.\end{aligned} \tag{6.1.20}$$

Frame-independence of the components of X allows us to assign values to these components, which depend solely on the position $x \in M$ and not on the specific choice of local frame, *as long as this is a local section of* $\mathcal{Q}$. Of course, this is still not possible explicitly on a generic manifold. However, for example if there are trivial directions in M, i. e. topologically $M^D = \tilde{M}^d \times I_1 \times \ldots \times I_{D-d}$, where $I_1, \ldots, I_{D-d}$ are intervals with coordinates $r_1, \ldots, r_{D-d}$, we could put $(\hat{e}^* X)_\mu = (\hat{e}^* X)_\mu(r_1, \ldots, r_{D-d})$.

In particular, this applies to the conical 6-manifolds, which we constructed in chapter 5. Therefore, we now specialize to manifolds of that type. That is, we consider 6-dimensional Riemannian manifolds of the type of a ϕ-cone over a 5-manifold, i. e.

$$\big(M^6,\, g\big) = \big(C_\phi(M^5),\, g\big) = \big(M^5 \times I,\, \phi^2\, g_5 + \mathrm{d}r^2\big). \tag{6.1.21}$$

We assume that M^6 is endowed with an $SU(2)$-structure $\mathcal{Q}$, which is a reduction of an $SU(3)$-structure $\mathcal{P}$. As the gauge principal bundle we choose

$$\mathfrak{B} = \mathcal{P}. \tag{6.1.22}$$

Thus, we have the following structure of bundle reductions:

$$\begin{array}{ccc} & \mathcal{Q} \times SU(2) & \\ \swarrow & & \searrow \\ SO\big(M^6, g\big) \times SO(6) & \longleftarrow & \mathcal{P} \times SU(3) \end{array} \tag{6.1.23}$$

All the maps in this diagram are inclusions as subbundles or subgroups, respectively, just as they occurred in chapter 5.

To fit the above considerations, we consider a connection $A \in \mathcal{C}(\mathcal{Q})$ on $\mathcal{Q}$, extend it to $\hat{A} \in \mathcal{C}(\mathcal{P})$ and perturb it by a frame-independent $X \in \Omega^1_{hor}(\mathcal{P}, \mathfrak{su}(3))^{(SU(3), Ad_{SU(3)})}$. Taking r to be the natural coordinate on the interval I, we restrict ourselves to the case, where

$$(\hat{e}^* X)_\mu = (\hat{e}^* X)_\mu\,(r) \tag{6.1.24}$$

depends on the cone direction only. We again take $\mu = (a, 5)$ and $\hat{\mu} = 1, \ldots, 6$. Furthermore, since X is frame-independent, we drop the sections e_σ in the remainder of this chapter wherever displaying them is not necessary, but keep in mind that all the local expressions are written down with respect to certain choices of local sections. We thus write

$$(\hat{e}^* X)_\mu = X_\mu, \quad (e_\sigma)_\mu = e_\mu \quad \text{and} \quad (\beta_\sigma)^\mu = \beta^\mu. \tag{6.1.25}$$

If we assume that $X_6 = 0$, the field strength of $\hat{A} + X$ reads

$$\begin{aligned} \hat{e}_\sigma{}^* F^{\hat{A}+X} = \lambda_* \circ e_\sigma{}^* F^A &+ \Big(\mathcal{L}_{\partial_r} X_\mu + T^\delta{}_{6\mu}\, X_\delta\Big) \otimes \mathrm{d}r \wedge \beta^\mu \\ &+ \frac{1}{2}\Big(T^\mu{}_{\nu\kappa}\, X_\mu + \big[X_\nu, X_\kappa\big]\Big) \otimes \beta^\nu \wedge \beta^\kappa \end{aligned}$$

$$
\begin{aligned}
&= \lambda_* \circ e_\sigma{}^* F^A \\
&\quad - \left(\mathcal{L}_{\partial_r} X_\mu + T^\delta_{\ 6\mu}\, X_\delta\right) \otimes \left(\beta^\mu \wedge \beta^6 - \frac{1}{2} N^\mu{}_{\nu\kappa}\, \beta^\nu \wedge \beta^\kappa\right) \\
&\quad + \frac{1}{2}\left(T^\mu{}_{\nu\kappa}\, X_\mu + \left[X_\nu, X_\kappa\right] - N^\mu{}_{\nu\kappa}\left(\mathcal{L}_{\partial_r} X_\mu + T^\delta_{\ 6\mu}\, X_\delta\right)\right) \otimes \beta^\nu \wedge \beta^\kappa .
\end{aligned} \tag{6.1.26}
$$

We have added and subtracted $\frac{1}{2} N^\mu{}_{\nu\kappa}\, \mathcal{L}_{\partial_r} X_\mu \otimes \beta^\nu \wedge \beta^\kappa$, wherein $N \in \Omega^2(M^6, TM^6)$. The additional term appearing in the second line above is intended to make the 2-form part of that line satisfy the instanton condition. This anticipates that for particular choices of N, the locally defined forms given by

$$
\beta^\mu \wedge \beta^6 - \frac{1}{2} N^\mu{}_{\nu\kappa}\, \beta^\nu \wedge \beta^\kappa \quad \forall\, \mu = 1, \dots, 5 \tag{6.1.27}
$$

will be instantons for the $SU(3)$-structure $\mathcal{P}$ on the respective 6-manifolds.
Finally, the splitting

$$
\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathrm{m} \tag{6.1.28}
$$

is $SU(2)$-invariant, as observed in chapter 3. From section 3.4 one can see that, as vector spaces, $\mathrm{m} \simeq \mathbb{R}^5$. Moreover, as representations of $SU(2)$ on $\mathbb{R}^5$,

$$
(Ad_{SU(3)})_{|SU(2)} = \rho_{|SU(2)}, \tag{6.1.29}
$$

where ρ is the standard representation of $GL(5, \mathbb{R}^D)$ on $\mathbb{R}^D$. Therefore,

$$
\begin{aligned}
TM^5 &= SO(M^5, g_5) \times_{(SO(5),\rho)} \mathbb{R}^5 \\
&= \mathcal{Q} \times_{(SU(2),\rho)} \mathbb{R}^5 \\
&= \mathcal{Q} \times_{(SU(2), Ad_{SU(3)})} \mathrm{m}.
\end{aligned} \tag{6.1.30}
$$

We take $\{I_i \,|\, i = 6, 7, 8\}$ to be a basis of $\mathfrak{su}(2)$ and $\{I_\mu \,|\, \mu = 1, \dots, 5\}$ to be a basis of m, such that $I_A = (I_\mu, I_i)$ is a basis of $\mathfrak{su}(3)$ that is adapted to the splitting $\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathrm{m}$. By $f_{AB}{}^C$ we denote the structure constants of $\mathfrak{su}(3)$ with respect to this basis.
Thereby, we have arrived at the statement which will be key to the instanton constructions in this chapter.

Proposition 6.1.4: *Let M^6 be the ϕ-cone over a 5-manifold M^5 with the following data:*

(1) M^6 carries an $SU(2)$-structure $\mathcal{Q}$ and an $SU(3)$-structure $\mathcal{P}$ satisfying the relations (5.1), as well as an $SU(3)$-structure $\mathcal{P}'$, which may differ from $\mathcal{P}$.

(2) $A \in \mathcal{C}(\mathcal{Q})$ is an instanton for $\mathcal{P}'$.

(3) There is an $N \in \Omega^2(M^6, TM^6)$ such that for $\mu = 1, \dots, 5$ the local forms $\beta^\mu \wedge \beta^6 - \frac{1}{2} N^\mu{}_{\nu\kappa}\, \beta^\nu \wedge \beta^\kappa$ satisfy the instanton condition induced by $\mathcal{P}'$.

(4) $X \in \Omega^1_{hor}(\mathcal{P}, \mathfrak{su}(3))^{(SU(3), Ad_{SU(3)})}$ *has frame-independent components with respect to sections of* $\mathcal{Q}$*, i. e.*

$$0 = \left[I_i, X_\mu\right] - \rho_*(I_i)^\nu{}_\mu X_\nu = \left[I_i, X_\mu\right] - f_{i\mu}{}^\nu X_\nu \quad \forall\, i = 6, 7, 8, \tag{6.1.31}$$

where, additionally, $X_6 = 0$*, and the* X_μ *depend on the cone direction only.*

(5) The local representations of the 2*-form*

$$\mathcal{N} \coloneqq \frac{1}{2}\, pr_{\mathfrak{su}(2)}\big(ad(X) \overset{\circ}{\wedge} (X)\big) \;\in \Omega^2_{hor}(\mathcal{P}, \mathfrak{su}(3))^{(SU(3), Ad_{SU(3)})} \tag{6.1.32}$$

satisfy the instanton condition induced by $\mathcal{P}'$*. The components of* $\mathcal{N}$ *are given by* $\mathcal{N}_{\mu\nu} = e^*\mathcal{N}\,(e_\mu, e_\nu)$ *for local sections* e *of* $\mathcal{Q}$*.*

Then, if X *satisfies the equations*

$$\left[X_\mu, X_\nu\right] + T^\kappa{}_{\mu\nu}\, X_\kappa - N^\kappa{}_{\mu\nu}\, \left(\mathcal{L}_{\partial_r} X_\kappa + T^\rho{}_{6\kappa}\, X_\rho\right) = \mathcal{N}_{\mu\nu}, \tag{6.1.33}$$

$\hat{A} + X \in \mathcal{C}(\mathcal{P})$ *is an instanton for* $\mathcal{P}'$*.*

Proof. Under the assumptions of the proposition, the field strength of $\hat{A} + X$ is given by (6.1.26). If A is an instanton itself, the first term in that equation satisfies the instanton condition. Due to (3), the second term satisfies that condition as well. All that is left is to require the third term to be an instanton, which is true if (6.1.33) and assumption (5) are satisfied. □

A few comments are in order. First, note that for any Lie group G with Lie algebra $\mathfrak{g}$ one can show (cf. section 4.3)

$$[Ad(g)(\xi_1), Ad(g)(\xi_2)] = Ad(g)\big([\xi_1, \xi_2]\big) \quad \forall\, g \in G,\ \xi_1, \xi_2 \in \mathfrak{g}. \tag{6.1.34}$$

This, together with the frame-independence of X with respect to $\mathcal{Q}$, implies that $\mathcal{N}$ satisfies the frame-independence condition for $\mathcal{Q}$ as well. With respect to local coframes β adapted to $\mathcal{Q}$ its coefficients read

$$\mathcal{N}_{\mu\nu} = \frac{1}{2}\, pr_{\mathfrak{su}(2)}\big([X_\mu, X_\nu]\big) \otimes \beta^\mu \wedge \beta^\nu = \frac{1}{2}\, f_{\kappa\lambda}{}^i\, X^\kappa_\mu\, X^\lambda_\nu\; I_i \otimes \beta^\mu \wedge \beta^\nu \tag{6.1.35}$$

In principle, one could drop assumption (5) and just require the coefficients of the last term in (6.1.26) to be the coefficients of an instanton. The problem is that upon choosing $X_\mu : M^6 \to \mathrm{m}$, the commutator $[X_\mu, X_\nu]$ has components in $\mathfrak{su}(2)$ as well as in m. We try to cancel the contributions in m among the torsion and N-terms, thus leaving the part in $\mathfrak{su}(2)$ untouched. Therefore, we have to require that part to satisfy the instanton condition, which is precisely property (5) of the proposition. Finally, the first three assumptions are properties of the geometries we are dealing with. In order to find new instanton solutions on a geometry with these properties, we have to find a set $X_1, \ldots, X_5$, which satisfy (4), (5) and (6.1.33). This will be the concern of the remainder of this chapter.

6.2 Sasaki-Einstein Structures and the First Instantons

Let us now try to find geometries that satisfy the assumptions of proposition 6.1.4. From our elaborations in chapter 5 we already know that the conical $SU(3)$ 6-manifolds constructed there satisfy the first assumption. This section is devoted to assumption (2), namely to finding a connection on an $SU(2)$-structure $\mathcal{Q}$ on M^6, which is an instanton for the $SU(3)$-structure $\mathcal{P}'$ that defines the geometry of M^6. Note that we do not require $\mathcal{Q}$ to be contained in that $SU(3)$-structure, as we clarify at the end of this section.

The first part of this section consists of reproduction of results from [2], whose combination with our findings in chapter 4 will enable us to find connections as desired. In [2], the following notion is used:

Definition 6.2.1: *Let M^D be a manifold, and let μ, ν, ρ, λ run from 1 to D. Be ∇ a covariant derivative on TM stemming from a connection on $F(TM)$, and be $R \in \Omega^2(M, End(TM))$ its field strength tensor. We say that R has the* ***interchange symmetry*** *if*

$$R_{\mu\nu\,\rho\lambda} = R_{\rho\lambda\,\mu\nu} \quad \forall\, \mu, \nu, \rho, \lambda = 1, \ldots, D. \tag{6.2.1}$$

Let us explain the relevance of this property in the present context. Consider a G-structure $\mathcal{Q}$ on M, and let β be an adapted coframe. On a connection on TM the instanton condition (4.1.17) precisely requires the 2-form indices of R with respect to β to represent a matrix in $\mathfrak{g} \subset \mathfrak{so}(D)$. However, if ∇ even stems from a connection on $\mathcal{Q} \subset F(TM)$, the second pair of indices of R yields a matrix in $\mathfrak{g}$, and, thus, the interchange symmetry implies that the connection is an instanton. We highlight this as

Lemma 6.2.2: *Let $\mathcal{Q}$ be a G-structure on M and ∇ a covariant derivative on TM stemming from a connection on $\mathcal{Q}$.*
If the field strength tensor of ∇ has the interchange symmetry, ∇ is an instanton for $\mathcal{Q}$ in the sense of (4.1.17)*.*

Therefore, it is desirable to find such connections on TM. Here the following proposition taken from [2] is very helpful:

Proposition 6.2.3: *Let ∇^t stem from a connection on $SO(M,g)$, i. e. $\nabla^t g = 0$ for all t, with totally antisymmetric torsion $\flat_g(T^t) = t\,P$ for some $P \in \Omega^3(M)$ and real parameter t. Suppose that when $t = 1$, P is parallel, that is, $\nabla^1 P = 0$. Then R^{∇^t} has the interchange symmetry for all t.*

Proof. See proposition 3.1 of [2]. □

We, hence, aim at finding connections on $F(TM)$ compatible with the G-structure $\mathcal{Q}$ and having totally antisymmetric, parallel torsion. We know that these will be instantons for $\mathcal{Q}$ by proposition 6.2.3.

Let us come back to the special spaces constructed in chapter 5. Therefore, from now on we again take $a, b, \dots$ running from 1 to 4, $\mu, \nu, \dots$ from 1 to 5 and $\hat{\mu}, \hat{\nu}, \dots$ from 1 to 6.

First, let us consider a Sasaki-Einstein 5-manifold M^5 with $SU(2)$-structure $\mathcal{Q}$. As explained in [17], Sasaki-Einstein $SU(2)$-structures on 5-manifolds are defined by a Killing spinor. From this spinor, all the defining sections of $\mathcal{Q}$ can be deduced [2]. It has been shown in that reference that one can construct a covariant derivative ∇^P on TM with respect to which the Killing spinor is parallel. Thus, ∇^P originates from a connection Γ^P on $\mathcal{Q}$. The torsion of Γ^P is given by [2,12]

$$T_a = \frac{3}{4}\, P_{a\mu\nu}\, \beta^\mu \wedge \beta^\nu \quad \text{and} \quad T_5 = P_{5\mu\nu}\, \beta^\mu \wedge \beta^\nu, \tag{6.2.2}$$

where we used

$$P \coloneqq \eta \wedge \omega^3. \tag{6.2.3}$$

Thus, Γ^P is a connection on $\mathcal{Q}$, but it does not have totally antisymmetric torsion. Note that, although this connection preserves the Sasakian structure of M^5, it does not coincide with the characteristic connection of a Sasakian manifold one could have constructed from proposition 3.3.5 (see also [28]). The latter is a connection belonging to the $U(2)$-structure that is the Sasakian structure, but fails to be an $SU(2)$-connection on $\mathcal{Q}$ (cf. [2]).

Consider a deformation of the Sasaki-Einstein $SU(2)$-structure to another $SU(2)$-structure $\mathcal{Q}_\varrho$, which is defined by $(\eta,\, \varrho^2\, \omega^\alpha)$, where $\varrho \in \mathbb{R}_+$ is taken to be constant. We have already seen in section 4.3 that such deformations stem from transformations $(\beta^a,\, \beta^5) \mapsto (\varrho\, \beta^a,\, \beta^5)$. As $\varrho \in \mathbb{R}_+$ is constant on M^5, we directly see that Γ^P still preserves η and $\varrho^2\, \omega^\alpha$ for $\alpha = 1, 2, 3$. Hence, Γ^P is a connection on $\mathcal{Q}_\varrho$ for all values of ϱ. As this transformation changes the metric, it has an effect on the Levi-Civita connection and thereby on the torsion of Γ^P. It turns out that this is completely antisymmetric and proportional to P for [2,12]

$$\varrho = \varrho_0 = \frac{2}{\sqrt{3}}. \tag{6.2.4}$$

As Γ^P preserves P, this totally antisymmetric torsion is parallel with respect to Γ^P in addition. Therefore, for this value of ϱ, R_{Γ^P} has the interchange symmetry due to proposition 6.2.3 and, thus, Γ^P is an instanton for $\mathcal{Q}_{\varrho_0}$ according to lemma 6.2.2. Nevertheless, since we made use of the Sasaki-Einstein $SU(2)$-structure $\mathcal{Q} = \mathcal{Q}_{\varrho=1}$

in the constructions in chapter 5, we need to find instantons for $\mathcal{Q}$, rather than $\mathcal{Q}_{\varrho_0}$. As we explained in section 4.2, the instanton condition (4.1.17) induced by $\mathcal{Q}$ is equivalent to

$$* \, (\eta \wedge F_A) = -F_A, \tag{6.2.5}$$

whence locally

$$W(\mathcal{Q}) = \mathrm{span}_{\mathbb{R}}\{\beta^{14} - \beta^{23},\ \beta^{13} + \beta^{24},\ \beta^{12} - \beta^{34}\}. \tag{6.2.6}$$

As this bundle is invariant under $\beta^a \mapsto \varrho\, \beta^a$, we directly infer

$$W(\mathcal{Q}) = W(\mathcal{Q}_\varrho) \quad \forall\, \varrho \in \mathbb{R}_+, \tag{6.2.7}$$

which we have shown in a very abstract manner in section 4.3 already. This means that Γ^P is an instanton for all $\mathcal{Q}_\varrho$, $\varrho \in \mathbb{R}_+$.
In summary, Γ^P is a connection on the principal bundle $\mathcal{Q}$ of the Sasaki-Einstein $SU(2)$-structure, and it satisfies the instanton condition induced by this structure.

We proceed to the extensions of the Sasaki-Einstein 5-manifolds to 6-dimensional spaces. First, consider $M^5 \times I$ with the direct-product metric $g = g_5 + \mathrm{d}r^2$. As we argued in section 5.1, $\mathcal{Q}$ on M^5 directly lifts to an $SU(2)$-structure $\hat{\mathcal{Q}}$ on $M^5 \times I$. Recall that we refer to this lift of the constant family $\mathcal{Q}_r = \mathcal{Q}$ to $M^5 \times I$ as the *pushforward* $SU(2)$-structure on $M^5 \times I$. Furthermore, recall that

$$SU(2) \hookrightarrow SO(5) \hookrightarrow SO(6), \tag{6.2.8}$$

where $SO(5)$ is embedded into $SO(6)$ as upper-left block matrices. The adjoint bundle $Ad(\hat{\mathcal{Q}})$ consists of elements of the form $[q, \frac{1}{2}\, \sigma_{\mu\nu}\, \tilde{E}^{\mu\nu}]$, where

$$\tilde{E}^{\mu\nu} = \begin{bmatrix} E^{\mu\nu} & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{so}(6), \tag{6.2.9}$$

with $\{E^{\mu\nu} \,|\, 1 \leqslant \mu < \nu \leqslant 5\}$ the standard basis of $\mathfrak{so}(5)$ as in section 4.1, and $\sigma_{\mu\nu}$ is such that $\frac{1}{2}\, \sigma_{\mu\nu}\, \tilde{E}^{\mu\nu} \in \mathfrak{su}(2) \subset \mathfrak{so}(6)$. These components $\sigma_{\mu\nu}$ are the same as for the 5-dimensional $SU(2)$-structure $\mathcal{Q}$. Thus, the components of the 2-forms in $W(\hat{\mathcal{Q}})$ are the same as in five dimensions as well, whence locally

$$W(\hat{\mathcal{Q}}) = \mathrm{span}_{\mathbb{R}}\big\{\beta^{14} - \beta^{23},\ \beta^{13} + \beta^{24},\ \beta^{12} - \beta^{34}\big\}. \tag{6.2.10}$$

Here the $\beta^{\hat{\mu}}$ are coframes on $M^5 \times I$ adapted to the $SU(2)$-structure $\hat{\mathcal{Q}}$.
From this we infer that the lift of Γ^P to the direct product $M^5 \times I$ (or the pullback along the projection to the slices) is an instanton for $\hat{\mathcal{Q}}$. We denote the lift of Γ^P to the direct product again by Γ^P. This lift is a connection on $F(T(M^5 \times I))$, but it restricts to a connection on $\hat{\mathcal{Q}}$, i. e. $\Gamma^P \in \mathcal{C}(\hat{\mathcal{Q}})$.

The procedure employed in chapter 5 then was to apply deformations to the push-forward $SU(2)$-structure $\hat{\mathcal{Q}}$, just as considered in chapter 4. The deformations we used in chapter 5 can in general be decomposed into two parts. First, there is a transformation of the type

$$\mathcal{R} : M^5 \times I \to GL(6, \mathbb{R}), \quad \mathcal{R}(x, r) = \begin{bmatrix} \phi(x,r)\, \mathbb{1}_5 & \\ & 1 \end{bmatrix} \tag{6.2.11}$$

that changes the metric from the direct product metric to the ϕ-cone metric. Second, we applied transformations

$$\mathcal{T} : M^5 \times I \to SO(6, \mathbb{R}), \tag{6.2.12}$$

which leave the metric invariant, but still deform the $SU(2)$-structure.
The deformations we employed in chapter 5 deformed the pushforward $SU(2)$-structure $\hat{\mathcal{Q}}$ on $M^5 \times I$ into other $SU(2)$-structures $\mathcal{Q}' = R_h\, \mathcal{Q}$. Therefore, proposition 4.3.1 tells us that the maps inducing these transformations satisfy

$$\mathcal{R}, \mathcal{T} : M^5 \times I \to N_{GL(6,\mathbb{R})}(SU(2)). \tag{6.2.13}$$

Furthermore, as $\mathcal{R}$ is proportional to the identity on $\beta^1, \ldots, \beta^5$, we can infer that

$$\Phi_{\mathcal{R}} : \mathfrak{so}(D) \to \mathfrak{so}(D), \quad \mu_{ab}\, E^{ab} \;\mapsto\; \mathcal{R}^a{}_i\, \mu_{ab}\, \mathcal{R}^b{}_j\, E^{ij} \tag{6.2.14}$$

leaves the subspace $\mathfrak{su}(2) \subset \mathfrak{so}(6)$ invariant. This does also hold true for $\mathcal{T}$, as this is $SO(6)$-valued (see proof of corollary 4.3.5). Corollary 4.3.5 then implies that all these $SU(2)$-structures on $M^5 \times I$ define the same instantons, as

$$W(\mathcal{Q}) = W(\mathcal{Q}') \tag{6.2.15}$$

for all these structures. Thus, the lift of Γ^P to $M^5 \times I$ an instanton for all $SU(2)$-structures $\mathcal{Q}'$ such that $(\hat{\mathcal{Q}}, \mathcal{Q}')$ satisfy the normal deformation property introduced in definition 4.3.2. In particular, this holds true for all the $SU(2)$-structures that we constructed in chapter 5.

Finally, we use proposition 5.1.1 to extend the $SU(2)$-structures considered above to $SU(3)$-structures on $C_\phi(M^5)$. Recall that $\mathcal{Q}' \subset \mathcal{P}'$, whence

$$W(\mathcal{Q}') \subset W(\mathcal{P}'). \tag{6.2.16}$$

This implies

Proposition 6.2.4: *Γ^P is an instanton in the sense of (4.1.17) for all the $SU(2)$-structures on $M^5 \times I$ obtained by lifting the Sasaki-Einstein $SU(2)$-structure of M^5 to $M^5 \times I$ and applying normal deformations leading to new $SU(2)$-structures.*

Moreover, Γ^P is an instanton for all the $SU(3)$-structures constructed as extensions of these $SU(2)$-structures by means of proposition 5.1.1.
In particular, this holds true for the $SU(2)$-structures and the $SU(3)$-structures on $C_\phi(M^5)$ constructed in chapter 5.

Therefore, on all these spaces, together with Γ^P, assumptions (1) and (2) of proposition 6.1.4 are satisfied.

This is a good point to review and summarize the geometric structure that we are concerned with. The frame bundle $F(T(M^5 \times I))$ provides the geometric background, where M^5 carries a Sasaki-Einstein $SU(2)$-structure. The embedding of M^5 induces the pushforward $SU(2)$-structure $\mathcal{Q}$ (we drop the hat now). By proposition 5.1.1 we know that this is contained in an $SU(3)$-structure $\mathcal{P}$. Furthermore, the lift of the Sasaki-Einstein canonical connection to the direct product is a connection on $\mathcal{Q}$, i.e. $\Gamma^P \in \mathcal{C}(\mathcal{Q})$. This is equivalent to the fact that if $e : U \to F(T(M^5 \times I))$ is a local section of $\mathcal{Q}$ or, in other words, a local frame adapted to the $SU(2)$-structure $\mathcal{Q}$, then

$$e^*\Gamma^P \in \Omega^1\big(U, \mathfrak{su}(2)\big). \tag{6.2.17}$$

We applied normal deformations to $\mathcal{Q}$ induced by $h : M^5 \times I \to N_{GL(6,\mathbb{R})}(SU(2))$, thus constructing a second $SU(2)$-structure $\mathcal{Q}' = R_h\,\mathcal{Q}$ from $\mathcal{Q}$. It is important to note that, in general, the restriction of Γ^P to the principal subbundle $\mathcal{Q}'$ of $F(T(M^5 \times I))$ is *not a connection on* $\mathcal{Q}'$ (cf. section 2.2). To illustrate this, consider

$$R_h \circ e : U \to \mathcal{Q}',\ x \mapsto R_{h(x)}\, e(x), \tag{6.2.18}$$

which is a local section of $\mathcal{Q}'$ by construction. By the properties of connection 1-forms, we have

$$(R_h \circ e)^*\Gamma^P = Ad(h^{-1}) \circ \Gamma^P + h^*\mu_{GL(6,\mathbb{R})}. \tag{6.2.19}$$

As h is $N_{GL(6,\mathbb{R})}(SU(2))$-valued, the homogeneous term is still a locally defined $\mathfrak{su}(2)$-valued 1-form on U. The inhomogeneous term, on the contrary, may take values anywhere in the Lie algebra of the normalizer of $SU(2)$ in $GL(6,\mathbb{R})$. Hence, in general, we can use Γ^P as an $\mathfrak{su}(2)$-valued connection on $\mathcal{Q}$ only. In section 4.3 we constructed a bijection $f_{\mathcal{Q},h}$ from $\mathcal{C}(\mathcal{Q})$ to $\mathcal{C}(\mathcal{Q}')$, and we could apply it here in order to obtain connections on $\mathcal{Q}'$. The map $f_{\mathcal{Q},h}$ takes a particularly simple form if h is constant, since in this case the inhomogeneous term is absent. In this case, $f_{\mathcal{Q},h}$ is merely the identity and connections on $\mathcal{Q}$ are connections on $\mathcal{Q}'$ as well. We will encounter such a situation in section 6.6.
Another particularly interesting case arises if h is further restricted to take values in the centralizer of $SU(2)$ in $GL(6,\mathbb{R})$, as we will consider in section 6.4.2.
Finally, $\mathcal{Q}'$ extends to an $SU(3)$-structure as well. This is the $SU(3)$-structure that

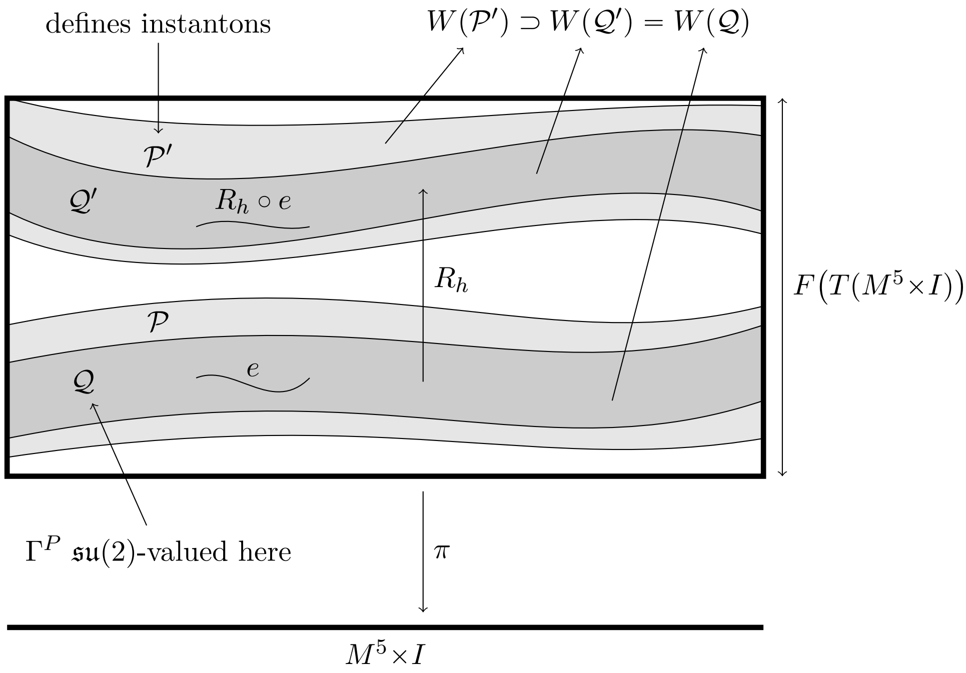

Figure 6.1.: Geometric structures on $F\big(T(M^5 \times I)\big)$

we use for the geometry of $M^5 \times I$ and, in particular, to define the instanton condition. The choice of the gauge principal bundle is independent of the choice of the $SU(3)$-structure. Nevertheless, we have to make sure that the sections e_σ that we used in section 6.1 are sections of the very same principal $SU(2)$-bundle which the instanton we are going to extend is an connection on. This is necessary since we explicitly used this property in the derivation of proposition 6.1.4, especially when we compared the pullback of the connection form with the local components that we used in the Maurer-Cartan identity. The latter have to be taken with respect to the same frame that we used as a section to pull the connection form back with. The geometric structures on the frame bundle of $F(T(M^5 \times I))$ are illustrated in figure 6.1.

We finish this section with a remark on instanton conditions on 6-manifolds with an $SU(3)$-structure $\mathcal{P}$ defined by (g, ω, Ω). As argued in section 4.2, the condition that the 2-form part of F_A takes values in $W(\mathcal{P})$ is equivalent to $*(\omega \wedge F_A) = -F_A$. A straightforward computation shows that this in turn is equivalent to

$$\Omega \wedge F_A = 0 \quad \text{and} \quad \omega \wedge \omega \wedge F_A = 0. \tag{6.2.20}$$

Applying the exterior covariant differential to the first equation as well as using the

Bianchi identity and $\Omega \wedge F_A = 0$ implies

$$\mathrm{d}\Omega \wedge F_A = \Big(\big(W_1^+ + i\, W_1^-\big)\, \omega \wedge \omega + \big(W_2^+ + i\, W_2^-\big) \wedge \omega \Big) \wedge F_A = 0. \tag{6.2.21}$$

Therefore, solutions to (6.2.20) automatically satisfy

$$\big(W_2^+ + i\, W_2^-\big) \wedge \omega \wedge F_A = 0. \tag{6.2.22}$$

On ordinary 2-forms $\sigma \in \Omega^2(M)$, this looks like an additional constraint for $W_2 \neq 0$. However, as for field strengths of connections there is the additional Bianchi identity, this is just a consequence of the instanton condition (6.2.20). Differentiating the second equation does not yield anything new, since both its sides are six-forms already.

Nevertheless, we could weaken the instanton condition to the requirement

$$\Omega \wedge F_A = 0, \tag{6.2.23}$$

which would imply

$$\Big(\big(W_1^+ + i\, W_1^-\big)\, \omega \wedge \omega + \big(W_2^+ + i\, W_2^-\big) \wedge \omega \Big) \wedge F_A = 0. \tag{6.2.24}$$

We might then define a different instanton bundle by

$$\widetilde{W}(\mathcal{P}) = \big\{ \sigma \in \Omega^2(M) \,\big|\, \Omega \wedge \sigma = 0,\ \big((W_1^+ + i\, W_1^-)\, \omega \wedge \omega + (W_2^+ + i\, W_2^-) \wedge \omega \big) \wedge \sigma = 0 \big\}. \tag{6.2.25}$$

Generically, the imposition of $\omega \wedge \omega \wedge F_A = 0$ would be an additional constraint on A and $\widetilde{W}(\mathcal{P}) \neq W(\mathcal{P})$. For $W_1 \neq 0$ and $W_2 = 0$ using (6.2.24) reproduces the original instanton condition (4.1.17).

6.3 Instantons on Kähler-Torsion Sine-Cones

Having developed the background for certain constructions of instantons very generally in the previous two sections, we now consider the explicit case of the Kähler-torsion sine-cones constructed in section 5.2.

Recall that we had

$$\big(C_\phi(M^5),\, g\big) = \Big(M^5 \times (0, \Lambda\, \pi),\ \Lambda^2 \sin\Big(\frac{r}{\Lambda}\Big)^2 g_5 + \mathrm{d}r^2 \Big). \tag{6.3.1}$$

For the metric we can compute

$$\begin{aligned} g &= \Lambda^2 \sin\Big(\frac{r}{\Lambda}\Big)^2 g_5 + \mathrm{d}r^2 = \Lambda^2 \sin\Big(\frac{r}{\Lambda}\Big)^2 \left(g_5 + \Big(\frac{\mathrm{d}r}{\Lambda \sin(\frac{r}{\Lambda})} \Big)^2 \right) \\ &= \Lambda^2 \sin\Big(\frac{r}{\Lambda}\Big)^2 \Big(g_5 + \big(\mathrm{d}\tau(r)\big)^2 \Big), \end{aligned} \tag{6.3.2}$$

where

$$\tau(r) = \log\left(2\Lambda\,\tan\left(\frac{r}{2\Lambda}\right)\right) \quad \Leftrightarrow \quad r(\tau) = 2\Lambda\,\arctan\left(\frac{e^\tau}{2\Lambda}\right). \tag{6.3.3}$$

This implies that there exists a conformal equivalence

$$f : M^5\times(0,\Lambda\,\pi) \to M^5\times\mathbb{R},\ f(x,r) = (x,\tau(r)), \tag{6.3.4}$$

between the sine-cone and the cylinder over M^5. We push the $SU(3)$-structure on the Kähler-torsion sine-cone forward to the cylinder along f and search for instantons for this $SU(3)$-structure on the cylinder. Their pullbacks to the sine-cone will then be instantons for the Kähler-torsion $SU(3)$-structure.

We aim at finding solutions to

$$*\,(\omega\wedge F_A) = -F_A. \tag{6.3.5}$$

Therefore, the instanton bundle on the cylinder can locally be written as

$$\begin{aligned} W(\mathcal{P}) = \mathrm{span}_{\mathbb{R}}\big\{&\beta^{14}-\beta^{23},\ \beta^{13}+\beta^{24},\ \beta^{12}-\beta^{34},\ \beta^{15}+\beta^{26},\\ &\beta^{16}-\beta^{25},\ \beta^{35}-\beta^{46},\ \beta^{45}+\beta^{36},\ \frac{1}{2}\beta^{12}+\frac{1}{2}\beta^{34}-\beta^{56}\big\}, \end{aligned} \tag{6.3.6}$$

from which we deduce the non-vanishing components of N appearing in (6.1.26):

$$N^5{}_{ab} = \frac{1}{2}\,\eta^3{}_{ab} \quad \text{and} \quad N^a{}_{5b} = -\eta^{3a}{}_b. \tag{6.3.7}$$

By the pushforward of the $SU(3)$-structure, these are the same on the sine-cone and on the cylinder in the respective coframes. N is a globally well-defined tensor field, for we can write it as

$$N = -\frac{1}{2}\,\xi\otimes\omega^3 - \eta\wedge J, \tag{6.3.8}$$

where $\xi = -e_5$ is the vector field dual to η. Thus, all geometric assumptions of proposition 6.1.4 are satisfied, and we are left to find suitable X_μ solving the remaining constraints and the matrix equations.
On the cylinder, Γ^P is a connection living on the pushforward $SU(2)$-structure $\mathcal{Q}$ and, in addition, an instanton for $\mathcal{P}$. Note that we have to perform our computations in a frame adapted to the $SU(2)$-structure on the cylinder, as explained in the previous section. That is, we use pushforwards of local frames adapted to the $SU(2)$-structure on M^5 to the cylinder, together with $e_6 = \partial_\tau$. The non-vanishing components of the torsion of Γ^P with respect to such a frame on the cylinder are just given by

$$T^5{}_{ab} = 2\,P^5{}_{ab} = -2\,\eta^3{}_{ab} \quad \text{and} \quad T^a{}_{5b} = \frac{3}{2}\,P^a{}_{5b} = \frac{3}{2}\,\eta^{3a}{}_b. \tag{6.3.9}$$

Recall that the matrix equations on the X_μ read

$$\begin{aligned} \big[I_i, X_\mu\big] &= f_{i\mu}{}^{\nu}\, X_\nu, \\ \big[X_\mu, X_\nu\big] &= -\, T^{\kappa}{}_{\mu\nu}\, X_\kappa + N^{\kappa}{}_{\mu\nu}\, \big(\mathcal{L}_{\partial_\tau} X_\kappa + T^{\rho}{}_{6\kappa}\, X_\rho\big) + \mathcal{N}_{\mu\nu}. \end{aligned} \tag{6.3.10}$$

These have to hold for all $i = 6,7,8$ and $\mu,\nu = 1,\dots,5$. Now we consider an ansatz which locally is of the form

$$\Gamma = \Gamma^P + X_\mu \otimes \beta^\mu. \tag{6.3.11}$$

Substituting N, T and $\mathcal{N}$, the matrix equations become

$$\begin{aligned} \big[I_i, X_\mu\big] &= f_{i\mu}{}^{\nu}\, X_\nu, \\ \big[X_5, X_b\big] &= -\,\frac{3}{2}\,\eta^{3a}{}_b\, X_a - \eta^{3a}{}_b\, \frac{\mathrm{d}}{\mathrm{d}\tau} X_a, \\ \big[X_a, X_b\big] &= 2\,\eta^3{}_{ab}\, X_5 + \frac{1}{2}\,\eta^3{}_{ab}\, \frac{\mathrm{d}}{\mathrm{d}\tau} X_5 + \mathcal{N}_{ab}. \end{aligned} \tag{6.3.12}$$

We have to make an ansatz for the X_μ. First, let us clarify which generators of $\mathfrak{su}(3)$ we are using here. We take the Killing-Cartan orthonormal basis from (3.4.11) and rescale it as in (3.4.13) with the choice

$$\gamma = -\frac{1}{3} \quad \text{and} \quad \delta = \pm\frac{1}{2\sqrt{3}}, \tag{6.3.13}$$

such that we obtain

$$T^5{}_{ab} = -2\,\eta^3{}_{ab} = -f_{ab}{}^5 \quad \text{and} \quad T^a{}_{5b} = \frac{3}{2}\,\eta^{3a}{}_b = -f_{5b}{}^a. \tag{6.3.14}$$

Then we employ the ansatz

$$X_a(\tau) = \psi(\tau)\, I_a,\;\; X_5(\tau) = \chi(\tau)\, I_5. \tag{6.3.15}$$

$\psi, \chi : \mathbb{R} \to \mathbb{R}$ are two real functions. From this we obtain

$$\mathcal{N}_{ab} = \psi^2\, f_{ab}{}^i\, I_i, \tag{6.3.16}$$

which can be seen to satisfy the instanton condition from (3.4.11) and (6.3.6). Moreover, the frame-independence condition is satisfied, since

$$[I_i, X_a] = \psi\, [I_i, I_a] = \psi\, f_{ia}{}^b\, I_b = f_{ia}{}^b\, X_b. \tag{6.3.17}$$

Hence, all assumptions of proposition 6.1.4 are indeed satisfied, and we are left to solve the matrix equations on the X_a. Upon inserting our ansatz they reduce to the coupled system of non-linear, ordinary differential equations given by

$$\dot{\psi} = \frac{3}{2}\,\psi\,(\chi - 1), \quad \dot{\chi} = 4\,(\psi^2 - \chi). \tag{6.3.18}$$

Here we denoted the derivative with respect to τ by the dot. These equations coincide with those obtained in [2] for the case of the $SU(3)$-structure on the cone over M^5 pushed forward to the cylinder. This is not surprising as the two $SU(3)$-structures induced on the cylinder by the conformal equivalences of the sine-cone considered here and the cone considered in [2] coincide. In principle, we have reproduced the reduction of the instanton equations carried out in [2].
Solutions to the above differential equations are given for example by

$$\big(\chi(\tau),\, \psi(\tau)\big) = (0,0), \tag{6.3.19}$$

$$\big(\chi(\tau),\, \psi(\tau))\big) = (1,\pm 1), \tag{6.3.20}$$

$$\big(\chi(\tau),\, \psi(\tau)\big) = (C\,\exp(-4\,\tau),\, 0). \tag{6.3.21}$$

Here, $C \in \mathbb{R}$ is a constant of integration. The first solution corresponds to the unperturbed Γ^P, which must occur as a solution for consistency because, as we know already, Γ^P is an instanton.
The solution $(\chi,\psi) = (1,1)$ is shown to be the pullback to the cylinder of the Levi-Civita connection of the metric cone in [2]. Since the metric cone over a Sasaki-Einstein manifold is Calabi-Yau, it has holonomy $SU(3)$. Furthermore, the Levi-Civita connection always has the interchange symmetry, and if it is compatible with the $SU(3)$-structure, it thus is an instanton itself. Additionally, recall that the Kähler-torsion $SU(3)$-structure on the sine-cone coincides with the pullback of the Calabi-Yau structure from the metric cone. Therefore, as the instanton condition is invariant under conformal transformations, the pullback of the Levi-Civita connection is an instanton on the Kähler-torsion sine cone.
The solution $(\chi,\psi) = (1,-1)$ reflects the fact that the defining sections of any $SU(2)$-structure are invariant under flipping the signs of $e_1,\dots,e_4$ of all the bases of the original structure. Therefore, carrying out the computation in these other sections of the $SU(2)$-bundle $\mathcal{Q}$, we would have obtained the same solution $(1,1)$, which expressed in the original basis reads $(\chi,\psi) = (1,-1)$.
The gauge field configurations corresponding to the third solution solve the instanton equation, but will not be of finite action due to the exponential factor, whence they are of less physical interest.
Analytical solutions to the above differential equations on χ and ψ other than these have neither been found in [2] nor in this work. For a slightly more extensive treatment of this system including some remarks on numerical solutions, we refer the reader to [2] and [39]. In the latter, Kähler-torsion sine-cones have been constructed over Sasaki-Einstein manifolds of generic dimension. Furthermore, the instanton solutions we found here have been shown to admit a generalization to arbitrary dimensions, as well.
Note that the solutions $(\chi,\psi) = (0,0)$, $(1,\pm 1)$ correspond to constant X_μ. Hence, they correspond to lifts of connections living on M^5 to the cylinder, which are in-

stantons for the $SU(2)$-structure on M^5.
All these instantons are connections on a gauge bundle with gauge group $SU(3)$ over a 6-dimensional manifold, which carries a Killing spinor ϵ [12]. In particular, all of the solutions we obtained satisfy

$$\gamma(F_A)(\epsilon) = 0. \tag{6.3.22}$$

For this reason, they may well be valuable starting points for the construction of solutions to the heterotic supergravity equations (1.3.5) on spaces with conical singularities. Finding such solutions to the instanton equations has been the first step towards the heterotic supergravity configurations found in [2–4].

6.4 Instantons on Nearly Kähler Sine-Cones

6.4.1 Reduction for Pushforward SU(2)-Structure

We turn our attention to the nearly Kähler sine-cones of section 5.3. Here we have $W_1 \neq 0$ and $W_2 = 0$ (see definition 3.1.5). Hence the weakened version of the instanton condition coincides with the ordinary instanton condition (4.1.17) or, equivalently, $*(\omega \wedge F_A) = -F_A$.
As before, we denote the lift of the constant $SU(2)$-structure on M^5 to $M^5{\times}I$ by $\mathcal{Q}$, where $I = (0, \Lambda\,\pi)$. Coframes adapted to $\mathcal{Q}$ are denoted by $\beta^{\hat{\mu}}$. Recall from the constructions in sections 5.2 and 5.3 that, in order to obtain the nearly Kähler sine-cone from this $SU(2)$-structure on $M^5{\times}I$, we have to employ two transformations. We have to rescale the coframes with

$$\mathcal{R} : M^5{\times}I \to GL(6,\mathbb{R}), \quad \mathcal{R}(x,r) = \begin{bmatrix} \Lambda\, \sin\left(\frac{r}{\Lambda}\right) \mathbb{1}_5 & \\ & 1 \end{bmatrix} \tag{6.4.1}$$

and to rotate them via

$$\mathcal{T} : M^5{\times}I \to SO(6), \quad \mathcal{T}(x,r) = \exp\left(\frac{r}{2\Lambda}\,\eta^2\right). \tag{6.4.2}$$

We denote by $\beta_s^{\hat{\mu}}$ the images of the $\beta^{\hat{\mu}}$ under these transformations, such that all the defining sections of the deformed $SU(2)$-structure $\mathcal{Q}' = R_{\mathcal{T}^{-1}\mathcal{R}^{-1}}\,\mathcal{Q}$ take their standard components with respect to β_s. This does then hold true for the defining sections of the nearly Kähler $SU(3)$-structure $\mathcal{P}'$ that $\mathcal{Q}'$ extends to as well. Therefore, N has precisely the same components (6.3.7) as in the previous case, but this time with respect to the coframes β_s. Again, N is globally well-defined.
The instanton which we perturb by X is the lift of Γ^P to the direct product, just as in the previous section. From section 6.2 we know that this will be an instanton for the nearly Kähler $SU(3)$-structure on the sine-cone.

However, since Γ^P is $\mathfrak{su}(2)$-valued on $\mathcal{Q}$ rather than $\mathcal{Q}'$, we have to perform computations in the original coframes β (as explained in section 6.2, in particular see figure 6.1). In the matrix equations (6.1.33) we, therefore, have to use the components of the tensor field N with respect to the untransformed coframes β. These read

$$\begin{aligned} N^a{}_{5b} &= -\Lambda \sin\left(\frac{r}{\Lambda}\right)\left(\cos\left(\frac{r}{\Lambda}\right)\eta^{3a}{}_b + \sin\left(\frac{r}{\Lambda}\right)\eta^{1a}{}_b\right), \\ N^5{}_{ab} &= \frac{1}{2}\Lambda \sin\left(\frac{r}{\Lambda}\right)\left(\cos\left(\frac{r}{\Lambda}\right)\eta^3{}_{ab} + \sin\left(\frac{r}{\Lambda}\right)\eta^1{}_{ab}\right). \end{aligned} \tag{6.4.3}$$

The torsion of Γ^P has components as in (6.3.14) with respect to β. Employing the reduction procedure of section 6.1 to an ansatz which locally reads

$$\Gamma = \Gamma^P + X_\mu \otimes \beta^\mu, \tag{6.4.4}$$

we arrive at the following matrix equations for the $X_\mu = (e^* X)(e_\mu)$:

$$\begin{aligned} \left[I_i, X_\mu\right] &= f_{i\mu}{}^\nu X_\nu, \qquad (6.4.5) \\ \left[X_5, X_a\right] &= -\frac{3}{2}\eta^{3b}{}_a X_b - \Lambda \sin\left(\frac{r}{\Lambda}\right)\left(\cos\left(\frac{r}{\Lambda}\right)\eta^{3a}{}_b + \sin\left(\frac{r}{\Lambda}\right)\eta^{1a}{}_b\right)\frac{\mathrm{d}}{\mathrm{d}r}X_b, \\ \left[X_a, X_b\right] &= 2\,\eta^3{}_{ab} X_5 + \frac{1}{2}\Lambda \sin\left(\frac{r}{\Lambda}\right)\left(\cos\left(\frac{r}{\Lambda}\right)\eta^3{}_{ab} + \sin\left(\frac{r}{\Lambda}\right)\eta^1{}_{ab}\right)\frac{\mathrm{d}}{\mathrm{d}r}X_5. \end{aligned}$$

Note that in the large-volume limit $\Lambda \to \infty$, where the nearly Kähler $SU(3)$-structure on the sine-cone approaches the Calabi-Yau $SU(3)$-structure on the metric cone over M^5, these equations smoothly tend to the equations that we would have obtained by carrying out the reduction procedure starting from Γ^P on the Calabi-Yau cone. For the ansatz $X_a = \psi\, I_a$ and $X_5 = \chi\, I_5$ the only solutions are given by

$$(\chi, \psi) = (0, 0) \quad \text{and} \quad (\chi, \psi) = (1, \pm 1). \tag{6.4.6}$$

The gauge fields that these solutions correspond to are, of course, precisely the same configurations as the ones obtained in the last section as instantons for the Kähler-torsion $SU(3)$-structure. This is not surprising, for these gauge fields are lifts of instantons on M^5 to the direct product. Therefore, they are instantons for the pushforward $SU(2)$-structure $\mathcal{Q}$ on $M^5 \times I$. However, we explained in section 6.2 how all the normal deformations of $\mathcal{Q}$ that we considered in chapter 5 define the same instantons, i. e. $W(\mathcal{Q}) = W(\mathcal{Q}')$ for these $SU(2)$-structures. Thus, the gauge fields given by $\Gamma^P + \psi\, I_a \otimes \beta^a + \chi\, I_5 \otimes \beta^5$ with $(\chi, \psi) = (0, 0), (1, \pm 1)$ will be instantons for all the $SU(3)$-structures constructed in chapter 5.

As we thus expect these solutions to arise in every situation where we try to extend Γ^P, the more interesting question is whether there are solutions apart from these. In order to answer this question, we can pursue at least two different strategies.

First, we could try to employ more general ansätze for the X_μ, aiming to find solutions to the matrix equations that depend on the cone coordinate. Such solutions will be obtained in section 6.5. Second, we could try to find other gauge bundles than $\mathcal{Q}$ and $\mathcal{P}$. In the case of the geometries at hand, we are given at least one additional natural candidate, namely $\mathcal{Q}' = R_h\,\mathcal{Q}$ and its extension to a principal $SU(3)$-bundle $\mathcal{P}'$. This strategy we will follow in the next subsections.

6.4.2 Canonical Connection of the Nearly Kähler Sine-Cone

The reason that we used the extension $\mathcal{P}$ of the pushforward $SU(2)$-structure $\mathcal{Q}$ on $M^5 \times I$ as the gauge principal bundle was that we generically do not know an instanton on any other bundle that we could try to extend. Although in section 4.3 we constructed a bijection between $\mathcal{C}(\mathcal{Q})$ and $\mathcal{C}(\mathcal{Q}')$ for all normal deformations $\mathcal{Q}'$ of $\mathcal{Q}$, this map does, in general, not map instantons to instantons. Here, however, we appear to be in a more special situation that allows us to circumvent this problem by observing a simpler relation between $\mathcal{C}(\mathcal{Q})$ and $\mathcal{C}(\mathcal{Q}')$. The central element is the following lemma:

Lemma 6.4.1: *Let $(\mathfrak{B}, \pi, M, H)$ be a principal fiber bundle over M with a principal G-subbundle $(\mathcal{Q}, \pi, M, G)$, and let $(U_\sigma, s_\sigma)_{\sigma\in\Sigma}$ be an open covering of M by local sections of $\mathfrak{B}$. Consider a map $h : M \to C_H(G)$, $x \mapsto h(x)$, where $C_H(G) \subset H$ is the centralizer of G in H.*
Then we have

(1) $\mathcal{Q}' = R_h\,\mathcal{Q}$ is a principal G-subbundle of $\mathfrak{B}$, and $(U_\sigma, R_h \circ s_\sigma)_{\sigma\in\Sigma}$ is an open covering of M by local sections of $\mathcal{Q}'$. Furthermore, the transition maps of the coverings $(U_\sigma, s_\sigma)_{\sigma\in\Sigma}$ and $(U_\sigma, R_h \circ s_\sigma)_{\sigma\in\Sigma}$ coincide.

(2) The family $(U_\sigma, h^*\mu_H)_{\sigma\in\Sigma}$ is the local representation of a globally well-defined $\Xi \in \Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H,Ad_H)}$ with respect to $(U_\sigma, R_h \circ s_\sigma)_{\sigma\in\Sigma}$, i. e.

$$\big(R_h \circ s_\sigma\big)^*\,\Xi = h^*\mu_H, \tag{6.4.7}$$

where μ_H is the Maurer-Cartan form on H.

Proof. Ad (1): Since the centralizer of G in H is contained in the normalizer of G in H, we know from proposition 4.3.1 that $\mathcal{Q}' = R_h\,\mathcal{Q}$ is a principal G-subbundle of $\mathfrak{B}$.
Consider two local sections $s, s' : U \to \mathcal{Q}$ of $\mathcal{Q}$ with $s' = R_g \circ s$ for some transition map $g : U \to G$. Then we have

$$R_h \circ s' = R_h \circ R_g \circ s = R_{h^{-1}gh} \circ R_h \circ s = R_g \circ (R_h \circ s). \tag{6.4.8}$$

In the last identity we used that $h(x)$ is central for G for all $x \in M$. This shows (1).

Ad (2): Since $C_H(G) \subset H$ is a Lie subgroup of H and h takes values in $C_H(G)$ exclusively,

$$h^* \mu_H = h^* \mu_{C_H(G)}, \tag{6.4.9}$$

where now $\mu_{C_H(G)}$ is the Maurer-Cartan form on the Lie group $C_H(G)$. Note that $\mu_{C_H(G)}$ takes values in Lie$(C_H(G))$ only, and that Ad_H restricted to G acts trivially on this Lie subalgebra of $\mathfrak{h}$. Thus, for all $g \in G$,

$$h^* \mu_H = h^* \mu_{C_H(G)} = Ad_H(g^{-1}) \circ h^* \mu_{C_H(G)} = Ad_H(g^{-1}) \circ h^* \mu_H. \tag{6.4.10}$$

Note that a family of local $\mathfrak{h}$-valued 1-forms Ξ_σ is a representation of some globally defined $\Xi \in \Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H, Ad_H)}$ with respect to $(U_\sigma, R_h \circ s_\sigma)_{\sigma \in \Sigma}$ if and only if [18]

$$\Xi_\sigma = Ad_H({g_{\sigma\rho}}^{-1}) \circ \Xi_\rho \tag{6.4.11}$$

for $s_\sigma = R_{g_{\sigma\rho}} \circ s_\rho$ and $\sigma, \rho \in \Sigma$.
We set

$$\Xi_\sigma = h^* \mu_H \quad \forall\, \sigma \in \Sigma. \tag{6.4.12}$$

From (1) we know that $R_h \circ s_\sigma = R_{g_{\sigma\rho}} \circ (R_h \circ s_\rho)$, whence (6.4.10) implies

$$\begin{aligned} \Xi_\sigma &= h^* \mu_H = Ad_H({g_{\sigma\rho}}^{-1}) \circ h^* \mu_H \\ &= Ad_H({g_{\sigma\rho}}^{-1}) \circ \Xi_\rho. \end{aligned} \tag{6.4.13}$$

Therefore, there exists a globally well-defined $\Xi \in \Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H, Ad_H)}$ satisfying

$$(R_h \circ s_\sigma)^* \, \Xi = \Xi_\sigma = h^* \mu_H \quad \forall\, \sigma \in \Sigma, \tag{6.4.14}$$

whence we have shown (2). □

Lemma 6.4.1 has the following important corollary:

Corollary 6.4.2: *If in the above situation $A \in \mathcal{C}(\mathfrak{B})$ is a connection on $\mathfrak{B}$ that restricts to a connection on $\mathcal{Q}$, then*

$$\tilde{A} := A - \Xi \tag{6.4.15}$$

is a connection on $\mathfrak{B}$ that restricts to a connection on $\mathcal{Q}'$. Its local representation with respect to $R_h \circ s_\sigma$ is

$$(R_h \circ s_\sigma)^* \tilde{A} = {s_\sigma}^* A \quad \forall\, \sigma \in \Sigma, \tag{6.4.16}$$

since h takes values in $C_H(G)$, only. From this it follows that also the field strengths have the same components with respect to S_σ and $R_h \circ s_\sigma$,

$$(R_h \circ s_\sigma)^* F^{\tilde{A}} = {s_\sigma}^* F^A \quad \forall\, \sigma \in \Sigma. \tag{6.4.17}$$

Proof. The statement that $\tilde{A}$ is a connection on $\mathfrak{B}$ is a direct consequence of the fact that the inhomogeneous terms dropped in (6.4.16) are the local representations of the globally well-defined form $\Xi \in \Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H,Ad_H)}$ and that $\mathcal{C}(\mathfrak{B})$ is an affine vector space over $\Omega^1_{hor}(\mathfrak{B}, \mathfrak{h})^{(H,Ad_H)}$. As $h(x)$ is central for G for all $x \in M$ and as s^*A is $\mathfrak{g}$-valued, we also see from (6.4.16) that $\tilde{A}$ is $\mathfrak{g}$-valued on $\mathcal{Q}'$ and therefore restricts to a connection on $\mathcal{Q}'$.
We have

$$(R_h \circ s_\sigma)^* \tilde{A} = (R_h \circ s_\sigma)^* (A - \Xi)$$
$$= Ad_H(h^{-1}) \circ {s_\sigma}^* A + h^* \mu_H - h^* \mu_H \tag{6.4.18}$$
$$= {s_\sigma}^* A. \tag{6.4.19}$$

In the las equality we used that h is central for g. This directly implies

$$(R_h \circ s_\sigma)^* F^{\tilde{A}} = {s_\sigma}^* F^A \quad \forall\, \sigma \in \Sigma, \tag{6.4.20}$$

thus completing the proof. □

This yields a bijection from $\mathcal{C}(\mathcal{Q})$ to $\mathcal{C}(\mathcal{Q}')$, which may, in general, differ from the one constructed in section 4.3. Due to the coinciding local representations of the field strengths of $\tilde{A}$ and A, this bijection even maps instantons to instantons.
One crucial application of this lemma to our situation is the following: Consider the manifold $M^5 \times I$, where again $I = (0, \Lambda\, \pi)$. Let $\mathcal{Q}$ be the $SU(2)$-structure obtained as the lift of the constant family given by just the Sasaki-Einstein $SU(2)$-structure on M^5 to $M^5 \times I$ in the sense of proposition 5.1.1. This is a principal $SU(2)$-subbundle of the principal $GL(6, \mathbb{R})$-bundle $F(T(M^5 \times I))$ and we have the embeddings

$$SU(2) \hookrightarrow SO(5) \hookrightarrow SO(6) \hookrightarrow GL(6, \mathbb{R}). \tag{6.4.21}$$

Now consider the two transformations defined by $\mathcal{R}$ and $\mathcal{T}$ as introduced in the preceding subsection. The essential observation is that both are central for this embedding of $SU(2)$ into $GL(6, \mathbb{R})$. Regarding $\mathcal{R}$, this can be seen from the fact that $\mathcal{R}$ is proportional to the identity on $SO(5)$. Further, $\omega^2 = \frac{1}{2}\, {\eta^2}_{\mu\nu}\, \beta^\mu \wedge \beta^\nu$ is invariant under $SU(2)$, as it is a defining section for the $SU(2)$-structure on M^5. Since $\mathcal{T}$ is generated by η^2, it commutes with all of $SU(2)$ as well.
By corollary 6.4.2 this implies that if $(\Gamma^P)^\mu{}_\nu$ are the components of Γ^P with respect to β, then *the same components* taken with respect to $\beta_s = (\mathcal{T} \circ \mathcal{R})(\beta)$ define a connection which is compatible with the rescaled and rotated $SU(2)$-structure underlying the nearly Kähler $SU(3)$-structure. We denote this connection by $\Gamma_{\mathfrak{su}(2)}$.
Let us for a moment distinguish between components that are computed with respect to β and β_s, respectively, by endowing indices with respect to the former with a

tilde. That is, for $e_s = R_h \circ e$ we write $\beta = (\beta^{\tilde{\mu}})$ and $\beta_s = (\beta_s^{\mu})$. Then, the local representation of $\Gamma_{\mathfrak{su}(2)}$ with respect to e_s is given by

$$\Gamma_{\mathfrak{su}(2)}{}^{\mu}{}_{\nu} = (\mathcal{T} \circ \mathcal{R})^{\mu}{}_{\tilde{\rho}}\, (\Gamma^P)^{\tilde{\rho}}{}_{\tilde{\lambda}}\, (\mathcal{R}^{-1} \circ \mathcal{T}^{-1})^{\tilde{\lambda}}{}_{\nu} = (\Gamma^P)^{\tilde{\mu}}{}_{\tilde{\nu}}. \tag{6.4.22}$$

By construction it is a connection on the principal bundle $\mathcal{Q}'$.

Even more, we know that on every nearly Kähler manifold there exists a connection preserving the $SU(n)$ structure, namely the Bismut connection, or the canonical connection, of the nearly Kähler structure, which we already encountered in section 3.1. We will denote this connection by $\Gamma_{\mathfrak{su}(3)}$. Having anticipated the existence of this connection, we now compute its connection 1-form by means of the Maurer-Cartan identity. We have

$$\begin{aligned}
\mathrm{d}\beta_s^a &= \mathrm{d}\big((\mathcal{T} \circ \mathcal{R})^a{}_{\tilde{b}}\, \beta^{\tilde{b}}\big) \\
&= \mathrm{d}(\mathcal{T} \circ \mathcal{R})^a{}_{\tilde{b}} \wedge \beta^{\tilde{b}} + (\mathcal{T} \circ \mathcal{R})^a{}_{\tilde{b}}\, \mathrm{d}\beta^{\tilde{b}} \\
&= \mathrm{d}\Big(\Lambda \sin\Big(\frac{r}{\Lambda}\Big)\, \exp\Big(\frac{r}{2\Lambda}\,\eta^2\Big)^a{}_{\tilde{b}}\Big) \wedge \beta^{\tilde{b}} \\
&\quad + (\mathcal{T} \circ \mathcal{R})^a{}_{\tilde{b}}\, \Big(-(\Gamma^P)^{\tilde{b}}{}_{\tilde{c}} \wedge \beta^{\tilde{c}} + T^{\tilde{b}}\Big) \\
&= \frac{1}{\Lambda}\Big(\cot\Big(\frac{r}{\Lambda}\Big)\,\delta^a_b + \frac{1}{2}\,\eta^{2a}{}_b\Big)\mathrm{d}r \wedge \beta_s^b \\
&\quad - (\mathcal{T} \circ \mathcal{R})^a{}_{\tilde{b}}\, (\Gamma^P)^{\tilde{b}}{}_{\tilde{c}}\, (\mathcal{R}^{-1} \circ \mathcal{T}^{-1})^{\tilde{c}}{}_d \wedge \beta_s^d \\
&\quad + \Big(\frac{3}{2\Lambda}\,\cot\Big(\frac{r}{\Lambda}\Big)\eta^{3a}{}_b - \frac{3}{2\Lambda}\,\eta^{1a}{}_b\Big)\beta_s^5 \wedge \beta_s^a \\
&= -\Gamma_{\mathfrak{su}(2)}{}^a{}_b \wedge \beta_s^b - \frac{1}{\Lambda}\,\cot\Big(\frac{r}{\Lambda}\Big)\,\Big(\beta_s^a \wedge \beta_s^6 + \eta^{3a}{}_b\,\beta_s^b \wedge \beta_s^5\Big) \\
&\quad - \frac{1}{2\Lambda}\,\cot\Big(\frac{r}{\Lambda}\Big)\,\eta^{3a}{}_b\,\beta_s^b \wedge \beta_s^5 + \frac{1}{\Lambda}\,\eta^{1a}{}_b\,\beta_s^b \wedge \beta_s^5 \\
&\quad - \frac{1}{2\Lambda}\Big(\eta^{2a}{}_b\,\beta_s^b \wedge \beta_s^6 - \eta^{1a}{}_b\,\beta_s^b \wedge \beta_s^5\Big),
\end{aligned} \tag{6.4.23}$$

as well as

$$\begin{aligned}
\mathrm{d}\beta_s^5 &= \mathrm{d}\Big(\Lambda \sin\Big(\frac{r}{\Lambda}\Big)\,\beta^{\tilde{5}}\Big) \\
&= \frac{1}{\Lambda}\,\cot\Big(\frac{r}{\Lambda}\Big)\,\beta_s^6 \wedge \beta_s^5 - \Lambda \sin\Big(\frac{r}{\Lambda}\Big)\,\eta^3{}_{ab}\,\beta_s^a \wedge \beta_s^b \\
&= -\frac{1}{\Lambda}\,\cot\Big(\frac{r}{\Lambda}\Big)\,\Big(\beta_s^5 \wedge \beta_s^6 + \eta^3{}_{ab}\,\beta_s^a \wedge \beta_s^b\Big) + \frac{1}{\Lambda}\,\eta^1{}_{ab}\,\beta_s^a \wedge \beta_s^b,
\end{aligned} \tag{6.4.24}$$

and

$$\mathrm{d}\beta_s^6 = \mathrm{d}^2 r = 0. \tag{6.4.25}$$

Introducing the complex coframes

$$\theta^1_s = \beta^1_s + i\,\beta^2_s, \quad \theta^2_s = \beta^3_s + i\,\beta^4_s, \quad \theta^3_s = i\,(\beta^5_s + i\,\beta^6_s) \tag{6.4.26}$$

and $\varphi := \frac{r}{\Lambda}$, this yields

$$\mathrm{d}\begin{pmatrix}\theta^1_s\\ \theta^2_s\\ \theta^3_s\end{pmatrix} = -\underbrace{\begin{bmatrix} {\hat\Gamma_{\mathfrak{su}(2)}}^1{}_1 + \frac{i\cot(\varphi)}{2\Lambda}\beta^5_s & {\hat\Gamma_{\mathfrak{su}(2)}}^1{}_2 & -\frac{\cot(\varphi)}{\Lambda}\theta^1_s - \frac{1}{2\Lambda}\theta^{\bar 2}_s \\ {\hat\Gamma_{\mathfrak{su}(2)}}^2{}_1 & {\hat\Gamma_{\mathfrak{su}(2)}}^2{}_2 + \frac{i\cot(\varphi)}{2\Lambda}\beta^5_s & -\frac{\cot(\varphi)}{\Lambda}\theta^2_s + \frac{1}{2\Lambda}\theta^{\bar 1}_s \\ \frac{\cot(\varphi)}{\Lambda}\theta^{\bar 1}_s + \frac{1}{2\Lambda}\theta^2_s & \frac{\cot(\varphi)}{\Lambda}\theta^{\bar 2}_s - \frac{1}{2\Lambda}\theta^1_s & -\frac{i\cot(\varphi)}{\Lambda}\beta^5_s \end{bmatrix}}_{\text{canonical }\mathfrak{su}(3)\text{-connection }\Gamma_{\mathfrak{su}(3)}\text{ on the nearly Kähler sine-cone}} \wedge \begin{pmatrix}\theta^1_s\\ \theta^2_s\\ \theta^3_s\end{pmatrix}$$

$$\underbrace{-\frac{1}{\Lambda}\begin{pmatrix}\theta^{\bar 2}_s \wedge \theta^{\bar 3}_s\\ \theta^{\bar 3}_s \wedge \theta^{\bar 1}_s\\ \theta^{\bar 1}_s \wedge \theta^{\bar 2}_s\end{pmatrix}}_{\text{NK-torsion }\hat T}. \tag{6.4.27}$$

We put a hat over the components of a connection to indicate its representation with respect to the complex basis θ, rather than the real basis β.

First, note that, upon complex conjugation, the matrix of 1-forms in the first line is mapped to minus itself, whence it is from $\mathfrak{su}(3)$ as it is necessary for a connection on $\mathcal{P}'$. However, from corollary 6.4.2 we only know that $\Gamma_{\mathfrak{su}(2)}$ is a well-defined connection. That this also holds true for $\Gamma_{\mathfrak{su}(3)}$ becomes clear if we isolate the 1-form part and decompose it into the real basis, thus giving

$$\hat\Gamma_{\mathfrak{su}(3)} = \hat\Gamma_{\mathfrak{su}(2)} + B_\mu \otimes \beta^\mu_s, \tag{6.4.28}$$

where

$$B_1 = \frac{1}{2\Lambda}\begin{bmatrix} 0 & 0 & -2\cot(\varphi)\\ 0 & 0 & 1\\ 2\cot(\varphi) & -1 & 0\end{bmatrix}, \qquad B_2 = \frac{i}{2\Lambda}\begin{bmatrix} 0 & 0 & -2\cot(\varphi)\\ 0 & 0 & -1\\ -2\cot(\varphi) & -1 & 0\end{bmatrix}$$

$$B_3 = \frac{1}{2\Lambda}\begin{bmatrix} 0 & 0 & -1\\ 0 & 0 & -2\cot(\varphi)\\ 1 & 2\cot(\varphi) & 0\end{bmatrix}, \qquad B_4 = \frac{i}{2\Lambda}\begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & -2\cot(\varphi)\\ 1 & -2\cot(\varphi) & 0\end{bmatrix}$$

$$B_5 = \frac{i}{2\Lambda}\begin{bmatrix} \cot(\varphi) & 0 & 0\\ 0 & \cot(\varphi) & 0\\ 0 & 0 & -2\cot(\varphi)\end{bmatrix}. \tag{6.4.29}$$

Here one can check that the B_μ satisfy the frame-independence condition (6.1.12), whence $B_\mu \otimes \beta^\mu_s$ is the local representation of some $B \in \Omega^1_{hor}(\mathcal{P}, \mathfrak{su}(3))^{(SU(3), Ad_{SU(3)})}$. Therefore, $\Gamma_{\mathfrak{su}(3)}$ is a well-defined connection on $\mathcal{P}'$, and we have

$$\Gamma_{\mathfrak{su}(3)} = \Gamma_{\mathfrak{su}(2)} + B \ \in \mathcal{C}(\mathcal{P}'). \tag{6.4.30}$$

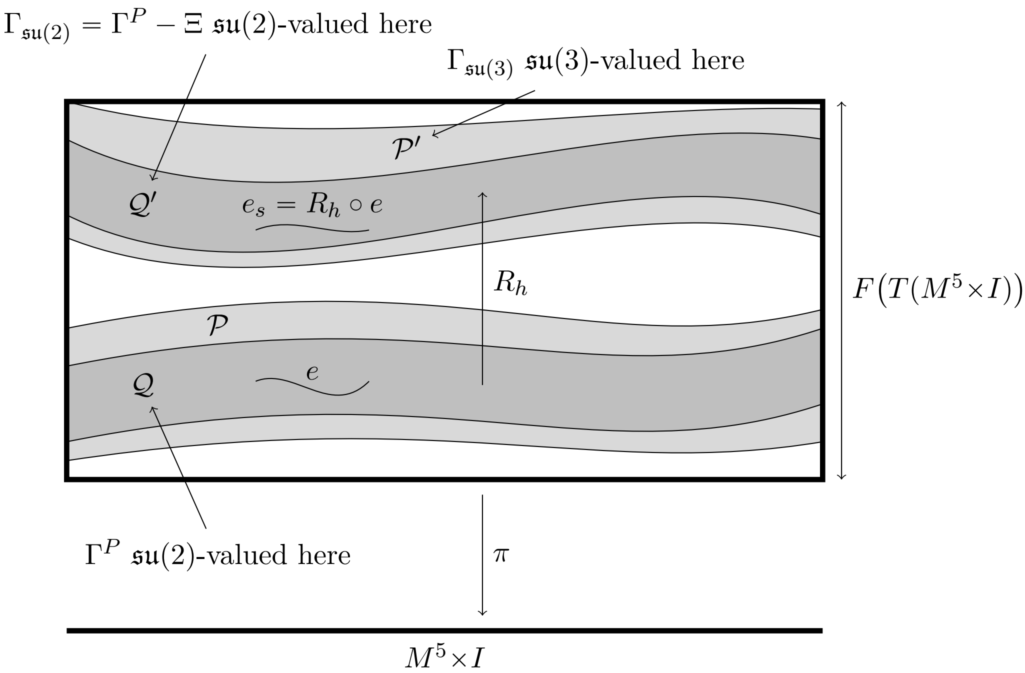

Figure 6.2.: Geometric structures on $F\big(T(M^5 \times I)\big)$ for nearly Kähler sine-cone

From (6.4.27) we see that the torsion of $\Gamma_{\mathfrak{su}(3)}$ is totally antisymmetric. Thus, we found a connection that preserves the nearly Kähler $SU(3)$-structure and has totally antisymmetric torsion. Due the uniqueness of such a connection [27, 28], this must then coincide with the canonical, or Bismut, connection of the nearly Kähler structure.
In fact, one can derive these explicit results without using the general statements of lemma 6.4.1 and its corollary 6.4.2. This route is taken in [40]. There we compute (6.4.27) without previously checking $\Gamma_{\mathfrak{su}(3)}$ or $\Gamma_{\mathfrak{su}(2)}$ to be a well-defined connection. We then observe that the torsion in (6.4.27) is invariant under $SU(3)$ and, in particular, that it transforms as a tensor. The Maurer-Cartan identity implies that $\Gamma_{\mathfrak{su}(3)}$ is a well-defined connection, and, finally, the frame-independence of the B_μ yields that $\Gamma_{\mathfrak{su}(2)}$ is a well-defined connection as well.
The torsion in (6.4.27) is invariant under $SU(3)$ and, therefore, parallel with respect to $\Gamma_{\mathfrak{su}(3)}$. As it is, in addition, totally antisymmetric, we infer from proposition 6.2.3 that $\Gamma_{\mathfrak{su}(3)}$ is an instanton on the nearly Kähler sine-cone over a Sasaki-Einstein 5-manifold. We again illustrate the geometric setup on $F(T(M^5 \times I))$, see figure 6.2.

We, thus, have seen an application of lemma 6.4.1 that allows us to construct connections on $SU(2)$-structures other than the pushforward $SU(2)$-structure $\mathcal{Q}$ very easily. In the above case,

$$pr_{\mathfrak{su}(2)}\Big(\mathrm{d}(\mathcal{T}\circ\mathcal{R})\,(\mathcal{T}\circ\mathcal{R})^{-1}\Big) = \frac{1}{\Lambda}\left(\cot\left(\frac{r}{\Lambda}\right)\delta^a_b + \frac{1}{2}\,\eta^{2a}{}_b\right)\mathrm{d}r = 0, \tag{6.4.31}$$

whence $\Gamma_{\mathfrak{su}(2)}$ coincides with the connection on $\mathcal{Q}'$ that we would have obtained upon application of proposition 4.3.3.
The particular example of $\Gamma_{\mathfrak{su}(2)}$ considered in this subsection is a connection living on $\mathcal{Q}'$, such that it preserves this deformed $SU(2)$-structure. Therefore, it will be compatible with its ambient nearly Kähler $SU(3)$-structure $\mathcal{P}'$ as well, and so will be all of the extensions of $\Gamma_{\mathfrak{su}(2)}$ to $\mathfrak{su}(3)$-connections living on $\mathcal{P}'$. In particular, this holds true for the extensions of $\Gamma_{\mathfrak{su}(2)}$ that we would use in the reduction procedure. Note that in the cone limit $\Lambda \to \infty$, the torsion in (6.4.27) tends to zero, such that $\Gamma_{\mathfrak{su}(3)}$ approaches the Levi-Civita connection of the Calabi-Yau metric cone over M^5.

From a string theoretic point of view, the canonical connection of the nearly Kähler structures might be of interest, as it might serve as the connection ∇^+ (cf. equation (1.3.1)) in explicit model building. Being a connection on the $SU(3)$-structure principal bundle $\mathcal{P}'$, it even preserves the Killing spinor which defines $\mathcal{P}'$, wherefore $\Gamma_{\mathfrak{su}(3)}$ already yields solutions to the first and the third equation in (1.3.1).

6.4.3 Reduction for Deformed SU(2)-Structure

In the last subsection, we constructed a connection $\Gamma_{\mathfrak{su}(2)}$ on the $SU(2)$-structure $\mathcal{Q}'$ This is the $SU(2)$-structure that extends to the nearly Kähler $SU(3)$-structure on the sine-cone over the Sasaki-Einstein M^5. From corollary 6.4.2 we know that the local representations of its field strength with respect to the β_s coincide with the local representations of the field strength of Γ^P expressed in the respective β. Now, since Γ^P is an instanton on the nearly Kähler sine-cone, so is $\Gamma_{\mathfrak{su}(2)}$.
Hence, the efforts of section 6.4.2 led to two additional explicit instantons for these nearly Kähler spaces. The canonical connection of the nearly Kähler structure has already been known, and has been known to be an instanton for this structure (see e. g. [2]). In contrast, $\Gamma_{\mathfrak{su}(2)}$ exists due to the very special geometry of the nearly Kähler sine-cone and appears to have been unknown previously.

We would like to revisit the reduction procedure for the instanton equations introduced in proposition 6.1.4. In section 6.4.1, we obtained instanton solutions on the nearly Kähler sine-cones using Γ^P. However, these were just the extensions of Γ^P found in section 6.3 already, and we argued that these extensions of Γ^P will be solutions in all the cases we consider here.
Nevertheless, we can also use $\Gamma_{\mathfrak{su}(2)}$ in this procedure, i. e. try an ansatz of the form

$$\Gamma = \Gamma_{\mathfrak{su}(2)} + X_\mu \otimes \beta^\mu_s. \tag{6.4.32}$$

Note that we have to use local frames that are sections of $\mathcal{Q}'$ now, in order that the formulas derived in section 6.1 apply directly. That is, we have to carry out the component computations with respect to β_s.
By the decomposition (6.4.28) and the fact that the B_μ satisfy the condition for frame-independence, we even are provided with an ansatz for the B_μ that we know will lead to non-trivial solutions of the resulting matrix equations.
First, we compute the torsion $T_{\mathfrak{su}(2)}$ of $\Gamma_{\mathfrak{su}(2)}$. We again indicate components that are taken with respect to the untransformed frames by a tilde over the respective indices. The Maurer-Cartan identity yields

$$\begin{aligned}
\mathrm{d}\beta_s^\mu &= \mathrm{d}\big((\mathcal{T}\circ\mathcal{R})^\mu{}_{\tilde\nu}\,\beta^{\tilde\nu}\big) \\
&= \mathrm{d}(\mathcal{T}\circ\mathcal{R})^\mu{}_{\tilde\nu}\wedge\beta^{\tilde\nu} + (\mathcal{T}\circ\mathcal{R})^\mu{}_{\tilde\nu}\,\mathrm{d}\beta^{\tilde\nu} \\
&= \mathrm{d}\Big(\Lambda\,\sin\Big(\frac{r}{\Lambda}\Big)\ \exp\Big(\frac{r}{2\Lambda}\,\eta^2\Big)^\mu{}_{\tilde\nu}\Big)\wedge\beta^{\tilde\nu} \\
&\quad + (\mathcal{T}\circ\mathcal{R})^\mu{}_{\tilde\nu}\ \Big(-(\Gamma^P)^{\tilde\nu}{}_{\tilde\rho}\wedge\beta^{\tilde\rho} + T^{\tilde\nu}\Big) \\
&= \ -\Gamma_{\mathfrak{su}(2)}{}^\mu{}_\nu\wedge\beta_s^\nu + \Bigg(\frac{1}{\Lambda}\Big(\cot\Big(\frac{r}{\Lambda}\Big)\,\delta^\mu_\nu + \frac{1}{2}\,\eta^{2\mu}{}_\nu\Big)\,\mathrm{d}r\wedge\beta_s^\nu + T^\mu\Bigg).
\end{aligned} \tag{6.4.33}$$

Thus, if we express both $T_{\mathfrak{su}(2)}$ and T with respect to β_s, we obtain

$$\begin{aligned}
&T_{\mathfrak{su}(2)}{}^\mu{}_{\nu\rho} = T^\mu{}_{\nu\rho}\,, \quad T_{\mathfrak{su}(2)}{}^6{}_{\nu\rho} = 0, \quad T_{\mathfrak{su}(2)}{}^6{}_{6\rho} = 0, \\
&T_{\mathfrak{su}(2)}{}^\mu{}_{6\rho} = \frac{1}{\Lambda}\,\Big(\cot\Big(\frac{r}{\Lambda}\Big)\,\delta^\mu_\rho + \frac{1}{2}\,\eta^{2\mu}{}_\rho\Big).
\end{aligned} \tag{6.4.34}$$

As we are still considering the same nearly Kähler $SU(3)$-structure as in section 6.4.1, the components of N are unaltered, and the general matrix equations for the perturbation X take the form

$$\begin{aligned}
\big[I_i, X_\mu\big] &= f_{i\mu}{}^\nu\,X_\nu, \\
\big[X_a, X_b\big] &= \frac{1}{2\Lambda}\,\eta^3{}_{ab}\,\Big(5\cot\Big(\frac{r}{\Lambda}\Big)\,X_5 + \Lambda\,\frac{\mathrm{d}}{\mathrm{d}r}X_5\Big) - \frac{2}{\Lambda}\,\eta^1{}_{ab}\,X_5 + \mathcal{N}_{ab}, \\
\big[X_5, X_a\big] &= \ -\frac{1}{2\Lambda}\,\eta^{3b}{}_a\,\Big(5\cot\Big(\frac{r}{\Lambda}\Big)\,X_b + 2\Lambda\,\frac{\mathrm{d}}{\mathrm{d}r}X_b\Big) + \frac{3}{2\Lambda}\,\eta^{1b}{}_a\,X_b - \frac{1}{2\Lambda}\,\eta^{3b}{}_a\,\eta^{2c}{}_b\,X_c.
\end{aligned} \tag{6.4.35}$$

Now we have already seen that the B_μ satisfy the first of these equations. This will still hold true for the choice

$$X_a(\varphi) = \psi(\varphi)\,B_a(\varphi) \quad \text{and} \quad X_5(\varphi) = \chi(\varphi)\,B_5(\varphi), \tag{6.4.36}$$

where we again make use of $\varphi \coloneqq \frac{r}{\Lambda}$. As one can check, the matrices X_μ yield

$$\mathcal{N}_{\mu\nu} = -\,\psi(\varphi)^2\,\frac{1 + 4\,\cot(\varphi)}{4\,\Lambda^2}\,f_{\mu\nu}{}^i\,I_i, \tag{6.4.37}$$

whence, for this choice of the X_μ, the 2-form part of $\mathcal{N}$ takes values in the instanton bundle $W(\mathcal{P}')$ only, just as desired. Inserting this ansatz for the X_μ into the matrix equations (6.4.35), we obtain, amongst certain differential equations, the algebraic constraint identity

$$\frac{\cot(\varphi)}{\Lambda^2}\big(\psi^2(\varphi) - \chi(\varphi)\big) = 0, \tag{6.4.38}$$

which reduces the differential equations to

$$\dot{\chi}(\varphi) = \dot{\psi}(\varphi) = 0 \quad \text{and} \quad \psi(\varphi)\big(\psi(\varphi)^2 - 1\big) = 0. \tag{6.4.39}$$

Here the dot denotes the derivative with respect to φ. The solutions to this system of equations are read off to be

$$(\psi,\, \chi) = (0, 0), \tag{6.4.40}$$

$$(\psi,\, \chi) = (1, 1), \tag{6.4.41}$$

$$(\psi,\, \chi) = (-1, 1). \tag{6.4.42}$$

First, $(\psi,\, \chi) = (0, 0)$ just reproduces $\Gamma_{\mathfrak{su}(2)}$. This solution must occur for consistency, since $\Gamma_{\mathfrak{su}(2)}$ was an instanton already.
Second, $(\psi,\, \chi) = (1, 1)$ reproduces the $\mathfrak{su}(3)$-valued instanton $\Gamma_{\mathfrak{su}(3)}$ of section 6.4.2. Consequently, we could have found this canonical connection of the nearly Kähler $SU(3)$-structure $\mathcal{P}'$ as an instanton extension of $\Gamma_{\mathfrak{su}(2)}$. This yields a fundamentally different proof that $\Gamma_{\mathfrak{su}(3)}$ is an instanton for $\mathcal{P}'$, without employing proposition 6.2.3, but using the much more explicit methods developed in section 6.1.
Third, the solution $(\psi,\, \chi) = (-1, 1)$ realizes another instanton on the nearly Kähler sine-cones we are considering. This, however, again arises due to the reflection symmetry in the β^a_s, which we had discussed in section 6.3 already.
Furthermore, note that if we express $\Gamma_{\mathfrak{su}(2)}$ with respect to β, it still has non-trivial components in the cone-direction, whence it is not a lift of a connection on M^5. That is, the connections we consider here really live on $C_{\sin}(M^5)$ rather than M^5. This is because $\mathcal{Q}'$ is a principal bundle that is not a pullback of a principal bundle over M^5. Generically, the right-action of h does not commute with the right-action of $SU(2)$, but here h takes values in the centralizer of $SU(2)$. Therefore, $\mathcal{Q}$ and $\mathcal{Q}'$ are isomorphic as principal bundles in the case at hand. Note that the map between connections on $\mathcal{Q}$ and on $\mathcal{Q}'$ constructed in lemma 6.4.1 and its corollary 6.4.2 is merely the pullback of connection 1-forms along this isomorphism of principal bundles. Nevertheless, the isomorphism stems from $M^5 \times I$ rather than M^5, whence the gauge field configurations really live on the 6-dimensional space.
The instantons we constructed here are isolated in the sense that none of them interpolates between two different solutions. In the limit $\Lambda \to \infty$, however, the strong algebraic constraint (6.4.38) becomes trivial. Hence, we might expect that there is

a richer space of solutions to the differential equations which the matrix equations reduce to in this limit. In particular, instanton solutions which interpolate between the instantons constructed from constant ψ and χ could be expected to occur.

Again, all of the above solutions are connections on a principal $SU(3)$-bundle which is defined by a Killing spinor. The gauge field configurations constructed here solve

$$\gamma(F_A)(\epsilon) = 0, \tag{6.4.43}$$

since we know that they satisfy the instanton condition in the sense of (4.1.17). Hence, all of them could prove useful in constructions of solutions to the field equations of heterotic supergravity (1.3.5) on spaces with conical singularities.

6.5 Interrelations and the Large-Volume Limit

Here we clarify the interrelations between the structures that we encountered on the since-cone over M^5. This will lead to a better understanding of the large-volume limit $\Lambda \to \infty$. Additionally, we observe a relation between instanton extensions of Γ^P and $\Gamma_{\mathfrak{su}(2)}$ on the nearly Kähler sine-cone that allows us to infer new solutions for both cases directly.

Let us once more consider the Kähler-torsion $SU(3)$-structure sine-cones over a Sasaki-Einstein 5-manifold as investigated in sections 5.2 and 6.3. There are several $SU(2)$- and $SU(3)$-structures that we encountered in this situation. First, there is the pushforward $SU(2)$-structure $\mathcal{Q}$ on $M^5{\times}I$, where $I = (0, \Lambda\pi)$. This is the lift of the constant family of $SU(2)$-structures consisting solely of the Sasaki-Einstein structure of M^5 to $M^5{\times}I$. Let $U \subset M^5$ be an open subset, and let $e : U{\times}I \to \mathcal{Q}$ be a local section of the principal fiber bundle $\mathcal{Q}$. With respect to this frame, the sine-cone metric reads

$$g_{\sin} = \Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 g_5 + \mathrm{d}r^2 = \Lambda^2 \sin\left(\frac{r}{\Lambda}\right)^2 \delta_{\mu\nu}\, \beta^\mu \otimes \beta^\nu + \beta^6 \otimes \beta^6. \tag{6.5.1}$$

Recall that the lift of Γ^P to the direct product $M^5{\times}I$ is a connection on $\mathcal{Q}$, i. e.

$$\Gamma^P \in \mathcal{C}(\mathcal{Q}). \tag{6.5.2}$$

We also considered the cylinder $M^5{\times}\mathbb{R}$. Here proposition 5.1.1 applies as well, and, therefore, we can lift the Sasaki-Einstein $SU(2)$-structure as a constant family to $M^5{\times}\mathbb{R}$, too. This $SU(2)$-structure we call $\mathcal{Q}_Z$. The lift of Γ^P to the direct product $M^5{\times}\mathbb{R}$ is a connection on $\mathcal{Q}_Z$ in complete analogy with the previous case. In order to distinguish the lifts to these two spaces, we denote the lift of Γ^P to the cylinder by Γ^P_Z. That is,

$$\Gamma^P_Z \in \mathcal{C}(\mathcal{Q}_Z). \tag{6.5.3}$$

We then deformed the pushforward $SU(2)$-structure $\mathcal{Q}$ on $M^5{\times}I$ by means of a right-action on $F(T(M^5{\times}I))$ induced by $\mathcal{R}$ as in (6.4.1). The image of this map we called $\mathcal{Q}' = R_{\mathcal{R}^{-1}}\,\mathcal{Q}$, and this was the $SU(2)$-structure underlying the Kähler-torsion $SU(3)$-structure. Moreover, this transformation maps e to a frame adapted to the sine-cone metric, i. e. writing $e' = R_{\mathcal{R}^{-1}} \circ e$ we obtain

$$g_{\sin} = \delta_{\hat{\mu}\hat{\nu}}\, \beta'^{\hat{\mu}} \otimes \beta'^{\hat{\nu}}. \tag{6.5.4}$$

Making use of lemma 6.4.1 and corollary 6.4.2, we see that there exists a connection on $\mathcal{Q}'$ given by

$$\begin{aligned} e'^*\Gamma_{\mathfrak{su}(2)} &= e'^*\Gamma^P - (\mathcal{R}^{-1})^*\mu_{GL(6,\mathbb{R})} \\ &= e^*\Gamma^P. \end{aligned} \tag{6.5.5}$$

Furthermore, there is the conformal equivalence

$$f : M^5{\times}I \to M^5{\times}\mathbb{R},\ (x,r) \mapsto (r, \tau(r)) \tag{6.5.6}$$

from the sine-cone to the cylinder over M^5 (cf. equations (6.3.3) and (6.3.4)). If g_Z is the cylinder metric on $M^5{\times}\mathbb{R}$, we have

$$f^* g_Z = \frac{1}{\Lambda^2\, \sin(\frac{r}{\Lambda})^2}\, g_{\sin}. \tag{6.5.7}$$

Let e_Z be a local section of $\mathcal{Q}_Z$, i. e. a local frame on the cylinder adapted to the pushforward $SU(2)$-structure, and let β_Z be its dual coframe. Then,

$$f^*\beta_Z^{\hat{\mu}} = \frac{1}{\Lambda\, \sin(\frac{r}{\Lambda})}\, \beta'^{\hat{\mu}} =: \tilde{\beta}^{\hat{\mu}}, \tag{6.5.8}$$

or, equivalently,

$$f_*(\tilde{e}_{\hat{\mu}}) \coloneqq f_*\Big(\Lambda\, \sin\Big(\frac{r}{\Lambda}\Big)\, e'_{\hat{\mu}}\Big) = e_{Z\,\hat{\mu}}. \tag{6.5.9}$$

The frames $\tilde{e}$ define yet another, intermediate, $SU(2)$-structure on M^5. We denote this new structure by $\widetilde{\mathcal{Q}}$. Another way of viewing $\widetilde{\mathcal{Q}}$ is as

$$\widetilde{\mathcal{Q}} = (f^{-1})_*(\mathcal{Q}_Z). \tag{6.5.10}$$

Local frames adapted to $\widetilde{\mathcal{Q}}$ and $\mathcal{Q}'$ are related by

$$e'_{\hat{\mu}} = \frac{1}{\Lambda\, \sin(\frac{r}{\Lambda})}\, \tilde{e}_{\hat{\mu}}. \tag{6.5.11}$$

In particular, defining

$$h : M^5{\times}I \to M^5{\times}I,\ (x,r) \mapsto \frac{1}{\Lambda\, \sin(\frac{r}{\Lambda})}\, \mathbb{1}_6 \tag{6.5.12}$$

we see that $(\widetilde{\mathcal{Q}}, \mathcal{Q}')$ satisfies the normal deformation property with respect to h. Note that h is central for $SU(2)$ and $SU(3)$ in $GL(6,\mathbb{R})$. Thus, the extensions $\widetilde{\mathcal{P}}$ and $\mathcal{P}'$ of $\widetilde{\mathcal{Q}}$ and $\mathcal{Q}'$, respectively, to $SU(3)$-structures satisfy the normal deformation property with respect to h as well.

The conformal equivalence, as an isomorphism of the principal fiber bundles $\mathcal{Q}_Z$ and $\widetilde{\mathcal{Q}}$, allows us to pullback Γ^P_Z from the cylinder to the sine-cone. Thereby, we obtain a connection on $\widetilde{\mathcal{Q}}$ that we denote by $f^*\Gamma^P_Z$. We have

$$\tilde{e}^*(f^*\Gamma^P_Z) = (f_*\tilde{e})^*\Gamma^P_Z = e_Z{}^*\Gamma^P_Z. \tag{6.5.13}$$

As h takes values in $C_{GL(6,\mathbb{R})}(SU(2))$, we can apply lemma 6.4.1 and corollary 6.4.2 to $f^*\Gamma^P_Z$ and h in order to obtain a connection $\widetilde{\Gamma}_{\mathfrak{su}(2)}$ on $\mathcal{Q}'$. The local representation of this connection reads

$$e'^*\widetilde{\Gamma}_{\mathfrak{su}(2)} = e'^*(f^*\Gamma^P_Z) - h^*\mu_{GL(6,\mathbb{R})} = \tilde{e}^*(f^*\Gamma^P_Z) = e_Z{}^*\Gamma^P_Z. \tag{6.5.14}$$

By construction, we can choose

$$e = (e_{M^5}, \mathrm{d}r) \quad \text{and} \quad e_Z = (e_{M^5}, \mathrm{d}\tau) \tag{6.5.15}$$

as local sections of $\mathcal{Q}$ on $M^5{\times}I$ and $\mathcal{Q}_Z$ on $M^5{\times}\mathbb{R}$, respectively, where e_{M^5} is a local frame on M^5 adapted to the Sasaki-Einstein $SU(2)$-structure. With respect to these sections, the lift of Γ^P to the direct products satisfies

$$(e^*\Gamma^P)_{|(x,r)} = (e_{M^5}{}^*\Gamma^P)_{|x} = (e_Z{}^*\Gamma^P_Z,)_{|(x,\tau)} \quad \forall\, r \in I,\ \tau \in \mathbb{R} \tag{6.5.16}$$

whence the connections induced from $\mathcal{Q}$ and $\mathcal{Q}_Z$ on $\mathcal{Q}'$ coincide, i. e.

$$\Gamma_{\mathfrak{su}(2)} = \widetilde{\Gamma}_{\mathfrak{su}(2)}. \tag{6.5.17}$$

Note that the pullback along f and the transformation induced by h are applicable to connections on the $SU(3)$ extensions of $\mathcal{Q}_Z$ and $\widetilde{\mathcal{Q}}$ as well, such that connections on $\mathcal{P}_Z$ induce connections on $\mathcal{P}'$.

The following diagram illustrates the interrelations between these geometric structures:

$$\begin{array}{ccccccc}
\Gamma^P & \longleftrightarrow & \Gamma_{\mathfrak{su}(2)} & \longleftrightarrow & f^*\Gamma^P_Z & \longleftrightarrow & \Gamma^P_Z \\
 & & & & & & \\
\mathcal{Q} & \xrightarrow{R_{\mathcal{R}^{-1}}} & \mathcal{Q}' & \xleftarrow{R_h} & \widetilde{\mathcal{Q}} & \xrightarrow{f_*} & \mathcal{Q}_Z \\
\downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\
\mathcal{P} & & \mathcal{P}' & \xleftarrow{R_h} & \widetilde{\mathcal{P}} & \xrightarrow{f_*} & \mathcal{P}_Z \\
 & & & & & & \\
 & & \mathcal{C}(\mathcal{P}') & \overset{1:1}{\longleftrightarrow} & \mathcal{C}(\widetilde{\mathcal{P}}) & \overset{1:1}{\longleftrightarrow} & \mathcal{C}(\mathcal{P}_Z)
\end{array} \tag{6.5.18}$$

Let us investigate the respective instanton conditions. The goal is to find instantons for the Kähler-torsion $SU(3)$-structure $\mathcal{P}'$ on the sine-cone over M^5. These are defined via the instanton bundle $W(\mathcal{P}') \subset \Lambda^2 T^*(M^5 \times I)$. As R_h is a rescaling of all the elements of the local frames $\tilde{e}$ (and, therefore, takes values in the normalizers of $SU(3)$ as well as $SU(2)$ in $GL(6, \mathbb{R})$), we have

$$W(\mathcal{P}') = W(\widetilde{\mathcal{P}}) \quad \text{and} \quad W(\mathcal{Q}') = W(\widetilde{\mathcal{Q}}). \tag{6.5.19}$$

In section 6.3 we tried to find instantons for $\mathcal{P}_Z$ on $M^5 \times \mathbb{R}$. Form $\mathcal{P}_Z = f_* \widetilde{\mathcal{P}}$ it follows that

$$f^*(W(\mathcal{P}_Z)) = W(\widetilde{\mathcal{P}}), \tag{6.5.20}$$

whence the pullbacks of $\mathcal{P}_Z$-instantons via f^*, or, more precisely, via the isomorphism of principal fiber bundles that it induces, are instantons for $\widetilde{\mathcal{P}}$ and hence as well for $\mathcal{P}'$.
In particular, assume that $A = \Gamma^P_Z + X$, i. e. locally

$$e_Z{}^* A = e_Z{}^* \Gamma^P_Z + X_\mu(\tau) \otimes \beta^\mu_Z, \tag{6.5.21}$$

is an instanton solution on the cylinder over M^5. In order to transport this solution to $\mathcal{Q}'$, we have to employ the pullback via f and rescale via h, obtaining a connection A' with the local representation

$$\begin{aligned} e'^* A' &= \tilde{e}^* \big(f^* (\Gamma^P_Z + X) \big) - h^* \mu_{GL(6,\mathbb{R})} \\ &= \Gamma_{\mathfrak{su}(2)} + \left(\frac{1}{\Lambda \, \sin(\frac{r}{\Lambda})} \; X_\mu\big(\tau(r)\big) \right) \otimes \beta'^\mu. \end{aligned} \tag{6.5.22}$$

Therefore, there is a direct one-to-one correspondence of solutions obtained in section 6.3 on the cylinder and solutions that we would have obtained upon applying the reduction procedure to $\Gamma_{\mathfrak{su}(2)}$ directly on the sine-cone. Upon using an ansatz $A' = \Gamma_{\mathfrak{su}(2)} + X$ with $X_a = \psi(r)\, I_a$ and $X_5 = \chi(r)\, I_5$ with respect to e' on the Kähler-torsion sine-cone we would, thus, have obtained the solutions

$$\big(\psi(r), \chi(r)\big) = (0, 0), \tag{6.5.23}$$

$$\big(\psi(r), \chi(r)\big) = \left(\frac{1}{\Lambda \, \sin\left(\frac{r}{\Lambda}\right)}, \frac{\pm 1}{\Lambda \, \sin\left(\frac{r}{\Lambda}\right)} \right), \tag{6.5.24}$$

$$\big(\psi(r), \chi(r)\big) = \left(0, \frac{C \, \cot\left(\frac{r}{2\Lambda}\right)^4}{16 \Lambda^5 \, \sin\left(\frac{r}{\Lambda}\right)} \right), \tag{6.5.25}$$

which are the transformations of the solutions (6.3.19). One can check, that these are indeed solutions to the matrix equations one had obtained on the Kähler-torsion sine-cone.

In particular, in the large-volume limit $\Lambda \to \infty$ this correspondence is preserved. Therefore, the instanton extensions of $\Gamma_{\mathfrak{su}(2)}$ on the sine-cone tend to the instantons that one obtains on the Calabi-Yau metric cone by transporting the solutions in section 6.3 from the cylinder to the cone directly.

On the nearly Kähler sine-cone, the definition of $\Gamma_{\mathfrak{su}(2)}$ did not only include the inhomogeneous term due to rescaling, but also the inhomogeneous term due to the rotation (6.4.2). In the large-volume limit, the rotation as well as its inhomogeneous contribution to $\Gamma_{\mathfrak{su}(2)}$ approach zero, and $B_\mu \to \frac{1}{r}\, I_\mu$. Hence, together with the relations pointed out in this section, one sees that in this limit the constructions of sections 6.3 and 6.4.3 coincide.
Another very interesting interrelation of the instanton extensions of connections on the tangent bundle that are related as in lemma 6.4.1 is the following. Denote the deformation that deforms $\mathcal{Q}$ into $\mathcal{Q}'$ by $h : M^5 \times I \to N_{GL(6,\mathbb{R})}(SU(2))$. The computations leading to (6.4.34) show that if $h = h(r)$ and $L(r)(e_6) = e_6$, we will have

$$T_{\mathfrak{su}(2)}{}^{\mu}{}_{6\nu} = (\mathrm{d}h^{-1}\, h)^{\mu}{}_{\nu} \tag{6.5.26}$$

(with respect to e'), while all other torsion components are unaltered, i. e. equal to those of Γ^P.
Recall that if $e : U \to \mathcal{Q}$ is a local section of $\mathcal{Q}$, and $e' = R_h \circ e : U \to \mathcal{Q}'$ is a local section of $\mathcal{Q}'$, then

$$e'_\mu = \rho(h)^{\nu}{}_{\mu}\, e_\nu = h^{\nu}{}_{\mu}\, e_\nu \tag{6.5.27}$$

and

$$\beta'^{\mu} = \left(\rho^{-1}(h)\right)_{\nu}{}^{\mu}\, \beta^{\nu} = (h^{-1})^{\mu}{}_{\nu}\, \beta^{\nu}. \tag{6.5.28}$$

Further, recall that $R_{\Gamma^P} = R_{\Gamma_{\mathfrak{su}(2)}}$ and that N is a tensor.
Consider an extension of $\Gamma_{\mathfrak{su}(2)}$ with local representation $X_\mu \otimes \beta'^{\mu}$. We have

$$X_\mu \otimes \beta'^{\mu} = (h^{-1})^{\mu}{}_{\nu}\, X_\mu \otimes \beta^{\mu}. \tag{6.5.29}$$

Now we insert this into the reduced expression for the field strength of Γ^P using (6.1.26). The crucial observation is that the inhomogeneous terms arising from $\mathcal{L}_{\partial_r} h$ reproduce the additional components $T_{\mathfrak{su}(2)}{}^{\mu}{}_{6\nu}$, and the factors of h^{-1} yield the transformations from the components with respect to e to those with respect to e'. Explicitly, marking tensor components with respect to e' be a prime,

$$\begin{aligned}
&\mathcal{L}_{\partial_r}\left((h^{-1})^{\mu}{}_{\nu}\, X_\mu\right) \otimes \left(\beta^{\nu} \wedge \beta^{6} - \frac{1}{2}\, N^{\nu}{}_{\sigma\kappa}\, \beta^{\sigma} \wedge \beta^{\kappa}\right) \\
&= \left(\left(\mathcal{L}_{\partial_r}(h^{-1})^{\mu}{}_{\nu}\right) X_\mu + (h^{-1})^{\mu}{}_{\nu} \left(\mathcal{L}_{\partial_r} X_\mu\right)\right) \otimes \left(\beta^{\nu} \wedge \beta^{6} - \frac{1}{2}\, N^{\nu}{}_{\sigma\kappa}\, \beta^{\sigma} \wedge \beta^{\kappa}\right) \\
&= \left(\left(\mathcal{L}_{\partial_r}(h^{-1})^{\mu}{}_{\lambda}\right) h^{\lambda}{}_{\nu}\, X_\mu + \mathcal{L}_{\partial_r} X_\nu\right) \otimes \left(\beta'^{\nu} \wedge \beta'^{6} - \frac{1}{2}\, N'^{\nu}{}_{\sigma\kappa}\, \beta'^{\sigma} \wedge \beta'^{\kappa}\right)
\end{aligned} \tag{6.5.30}$$

$$= \Big(\mathcal{L}_{\partial_r} X_\nu + {T'_{\mathfrak{su}(2)}}^{\delta}{}_{6\nu}\, X_\delta\Big) \otimes \Big(\beta'^\nu \wedge \beta'^6 - \frac{1}{2} N'^\nu{}_{\sigma\kappa}\, \beta'^\sigma \wedge \beta'^\kappa\Big),$$

and by the same calculation,

$$\begin{aligned} &\frac{1}{2}\Big(T^\mu{}_{\nu\kappa}\,(h^{-1})^\lambda{}_\mu X_\lambda + \big[(h^{-1})^\lambda{}_\nu X_\lambda, (h^{-1})^\sigma{}_\kappa X_\sigma\big] - N^\mu{}_{\nu\kappa}\,\mathcal{L}_{\partial_r}\big((h^{-1})^\lambda{}_\mu X_\lambda\big)\Big) \\ &\quad\otimes \beta^\nu \wedge \beta^\kappa \qquad (6.5.31) \\ &= \frac{1}{2}\Big(T'^\mu{}_{\nu\kappa} X_\mu + \big[X_\nu, X_\kappa\big] - N'^\mu{}_{\nu\kappa}\,\big(\mathcal{L}_{\partial_r} X_\mu + {T'_{\mathfrak{su}(2)}}^{\delta}{}_{6\mu}\, X_\delta\big)\Big) \otimes \beta'^\nu \wedge \beta'^\kappa. \end{aligned}$$

Thus, the local representations of the field strengths of $\Gamma^P + (h^{-1})^\mu{}_\nu X_\mu \otimes \beta^\mu$ with respect to e coincides with that of $\Gamma_{\mathfrak{su}(2)} + X_\mu \otimes \beta'^\mu$ with respect to e'. This implies that there is a one-to-one correspondence between instanton extensions of Γ^P and $\Gamma_{\mathfrak{su}(2)}$. However, note that, although the local representations of the extensions coincide, the 1-forms they correspond to live on different principal $SU(3)$-bundles $\mathcal{P}$ and $\mathcal{P}'$. They satisfy

$$e^* X = e'^* X' = (R_h \circ e)^* X' = Ad(h^{-1}) \circ e^* X'. \qquad (6.5.32)$$

Therefore, as long as h is not central for $SU(3)$,

$$X \neq X' \in \Omega^1\Big(F\big(T(M^5 \times I)\big), \mathfrak{gl}(6, \mathbb{R})\Big)^{(GL(6,\mathbb{R}),\rho)} \qquad (6.5.33)$$

Yet, their local representations coincide and thereby yield a simple identification of the instanton solutions.
Using this statement, we can deduce a solution to the system of matrix equations that arose in the reduction procedure as applied to $\Gamma_{\mathfrak{su}(2)}$ on the nearly Kähler sine-cone. We simply take the local representations of the extensions X of Γ^P from section 6.4.1 and use them as local representatives of an extension of $\Gamma_{\mathfrak{su}(2)}$ to $\mathcal{P}'$ rather than $\mathcal{P}$. By the above considerations we know that on $\mathcal{P}'$ there exists an instanton extension X' of $\Gamma_{\mathfrak{su}(2)}$ with local representations with respect to e' given by

$$(e'^* X')(e'_\mu) = X'_\mu = \big(\mathcal{R}^{-1}\, \mathcal{T}^{-1}\big)^\nu{}_\mu\, (e^* X)(e_\nu) \qquad (6.5.34)$$

in the terminology of section 6.4.1. Indeed, direct insertion of

$$\begin{aligned} X_1 &= \frac{\psi(r)}{\Lambda\, \sin(\frac{r}{\Lambda})}\, \left(\cos\left(\frac{r}{2\Lambda}\right) I_1 - \sin\left(\frac{r}{2\Lambda}\right) I_3\right), \\ X_2 &= \frac{\psi(r)}{\Lambda\, \sin(\frac{r}{\Lambda})}\, \left(\cos\left(\frac{r}{2\Lambda}\right) I_2 + \sin\left(\frac{r}{2\Lambda}\right) I_4\right), \\ X_3 &= \frac{\psi(r)}{\Lambda\, \sin(\frac{r}{\Lambda})}\, \left(\cos\left(\frac{r}{2\Lambda}\right) I_3 + \sin\left(\frac{r}{2\Lambda}\right) I_1\right), \end{aligned} \qquad (6.5.35)$$

$$X_4 = \frac{\psi(r)}{\Lambda\,\sin(\frac{r}{\Lambda})}\,\Big(\cos\Big(\frac{r}{2\Lambda}\Big)\,I_4 - \sin\Big(\frac{r}{2\Lambda}\Big)\,I_2\Big),$$
$$X_5 = \frac{\psi(r)}{\Lambda\,\sin(\frac{r}{\Lambda})}\,I_5$$

into the matrix equations (6.4.35) yields precisely the solutions

$$\big(\psi(r),\chi(r)\big) = (0,0),\ (\pm 1, 1) \tag{6.5.36}$$

in consistency with the results of section 6.4.1 and this section.
Conversely, the solutions for extensions of $\Gamma_{\mathfrak{su}(2)}$ in section 6.4.3 translate to instanton extensions of Γ^P. These have to be taken with respect to the local frames e adapted to $\mathcal{Q}$. They read

$$\begin{aligned}
X_1 &= \psi(r)\,\Big(\cos\Big(\frac{r}{2\Lambda}\Big)^3 I_1 - \sin\Big(\frac{r}{2\Lambda}\Big)^3 I_3\Big),\\
X_2 &= \psi(r)\,\Big(\cos\Big(\frac{r}{2\Lambda}\Big)^3 I_2 + \sin\Big(\frac{r}{2\Lambda}\Big)^3 I_4\Big),\\
X_3 &= \psi(r)\,\Big(\cos\Big(\frac{r}{2\Lambda}\Big)^3 I_3 + \sin\Big(\frac{r}{2\Lambda}\Big)^3 I_1\Big),\\
X_4 &= \psi(r)\,\Big(\cos\Big(\frac{r}{2\Lambda}\Big)^3 I_4 - \sin\Big(\frac{r}{2\Lambda}\Big)^3 I_2\Big),\\
X_5 &= \chi(r)\,\cos\Big(\frac{r}{\Lambda}\Big) I_5
\end{aligned} \tag{6.5.37}$$

and as well lead to the solutions

$$\big(\psi(r),\chi(r)\big) = (0,0),\ (\pm 1, 1). \tag{6.5.38}$$

One can check that the corresponding X_μ satisfy the matrix equations 6.4.5. Note that, although the functions that parametrize the ansatz are constant for these solutions, the coefficients X_μ depend on r in a non-trivial manner.

Thus, if the deformation h is central, one can transfer the solutions for an instanton perturbation to the deformed $SU(2)$-structure. Note that, while an instanton extension X may have constant local representations in the first case, its transformation to an instanton extension on $\mathcal{P}'$ may have much more intricate coefficients, as the above examples illustrate.

6.6 Instantons on Half-Flat Cylinders

To finish this chapter with, we turn our attention to the half-flat cylinders of section 5.4. Recall from proposition 5.4.2 that

$$W_2^- = \frac{4\varrho^2 - 3}{3\varrho}\,\big(\omega_z^3 - 2\,\mathrm{d}r \wedge \eta_z\big), \quad \text{and} \quad W_2^+ = 0 \tag{6.6.1}$$

for these spaces. According to our considerations at the end of section 6.2, we can now consider the instanton bundle to be either $W(\mathcal{P}')$, defined by the ordinary bundle construction as in chapter 4, or $\widetilde{W}(\mathcal{P}')$ as in (6.2.25), given by the requirement

$$\Omega \wedge \mu = 0 \tag{6.6.2}$$

and the derived equation

$$\begin{aligned} 0 = \mathrm{d}\Omega \wedge \mu &= \Big(\big(W_1^+ + i\, W_1^- \big)\, \omega \wedge \omega + \big(W_2^+ + i\, W_2^- \big) \wedge \omega \Big) \wedge \mu \\ &= i \left(\frac{3 + 2\varrho^2}{3\varrho}\, \omega \wedge \omega + \frac{4\varrho^2 - 3}{3\varrho} \left(\omega_z^3 - 2\, \mathrm{d}r \wedge \eta_z \right) \wedge \omega \right) \wedge \mu \\ &= i \left(2\, \varrho\, \omega_z^3 \wedge \omega_z^3 + \frac{3}{\varrho}\, \omega_z^3 \wedge \mathrm{d}r \wedge \eta_z \right) \wedge \mu \\ &= \frac{3i}{\varrho} \left(\frac{4\varrho^2}{3}\, \mu_{56} + \mu_{12} + \mu_{34} \right) \mathrm{vol}_g . \end{aligned} \tag{6.6.3}$$

Thus, we have

$$\begin{aligned} \widetilde{W}(\mathcal{P}') = \mathrm{span}_{\mathbb{R}} \Big\{ &\beta_z^{14} - \beta_z^{23},\ \beta_z^{13} + \beta_z^{24},\ \beta_z^{12} - \beta_z^{34},\ \beta_z^{15} + \beta_z^{26},\ \beta_z^{16} - \beta_z^{25}, \\ &\beta_z^{35} - \beta_z^{46},\ \beta_z^{45} + \beta_z^{36},\ \frac{1}{2}\, \beta_z^{12} + \frac{1}{2}\, \beta_z^{34} - \frac{3}{4\varrho^2}\, \beta_z^{56} \Big\}. \end{aligned} \tag{6.6.4}$$

Let us here consider this weaker version of the instanton condition, for it might allow for more interesting solutions. Hence, with respect to the coframes β_z,

$$N^b{}_{5a} = -\eta^{3b}{}_a \quad \text{and} \quad N^5{}_{ab} = \frac{2\,\varrho^2}{3}\, \eta^3{}_{ab}. \tag{6.6.5}$$

These are the components of the global tensor field

$$N = -\frac{2\,\varrho^2}{3}\, \xi \otimes \omega^3 - \eta \wedge J. \tag{6.6.6}$$

Note that W_2 vanishes for $\varrho = \frac{\sqrt{3}}{2}$, such that for this special value of ϱ the instanton bundle $\widetilde{W}(\mathcal{P})$ coincides with $W(\mathcal{P})$, and the components of N become the same as considered in the previous cases.
As in sections 6.3 and 6.4.1 we consider the lift of Γ^P to the direct product $M^5 \times I$. We have seen in section 6.2 that this is an instanton for the $SU(2)$-structure $\mathcal{Q}'$ underlying the half-flat $SU(3)$-structure on the cylinder. That is, the 2-form part of R_{Γ^P} lies in $W(\mathcal{Q}')$. This bundle is contained in $\widetilde{W}(\mathcal{P}')$ as we see from (6.2.6). Hence, Γ^P is an instanton for $\mathcal{P}'$, independently of which of the two notions of instantons we use.

The canonical procedure would be to employ the reduction of the instanton equations to Γ^P as a connection on $\mathcal{Q}$ and thus on its $SU(3)$ extension $\mathcal{P}$. However,

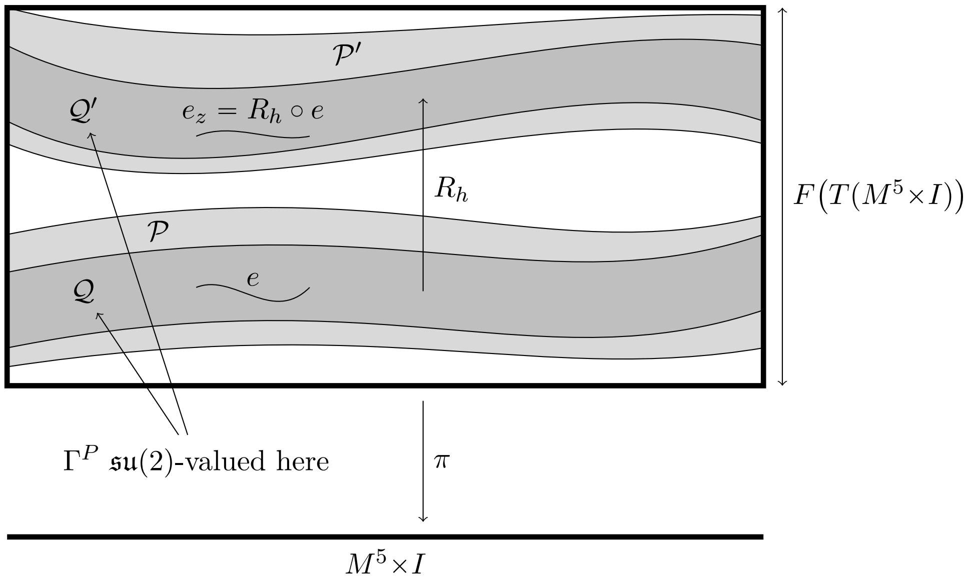

Figure 6.3.: Geometric structures on $F\big(T(M^5 \times I)\big)$ for half-flat cylinder

performing the reduction and trying to find solutions to the $X_\mu = (e^*X)(e_\mu)$, we only found the solutions of sections 6.3 and 6.4.1. As explained there, these are lifts from M^5 to $M^5 \times I$ of instantons for the Sasaki-Einstein $SU(2)$-structure on M^5 and, therefore, have to be present here as well. Nevertheless, more sophisticated ansätze might yield more interesting extensions of Γ^P to $\mathcal{P}$.
Here, as in section 6.4.2, it is valuable to take a closer look at the geometric construction of the half-flat $SU(3)$-structures $\mathcal{P}'$. From the introduction of these geometries in section 5.4 we recall that the normal deformations used in their construction are induced by *constant* maps $h : M^5 \times I \to N_{GL(6,\mathbb{R})}(SU(2))$. This is crucial here due to the following reason: Let $e : U \to \mathcal{Q}$ be a local section of $\mathcal{Q}$. Then $R_h \circ e : U \to \mathcal{Q}'$ is a local section of $\mathcal{Q}'$. Consider a connection $A \in \mathcal{C}(\mathcal{Q})$ on $\mathcal{Q}$. This can be extended to a connection on $F(T(M^5 \times I))$. As h is constant we have

$$(R_h \circ e)^* A = Ad\big(h^{-1}\big) \circ e^* A + h^* \mu_{GL(6,\mathbb{R})} = Ad\big(h^{-1}\big) \circ e^* A, \qquad (6.6.7)$$

and as we know that h takes values in the normalizer of $SU(2)$ in $GL(6, \mathbb{R})$, we infer that $(R_h \circ e)^* A$ is $\mathfrak{su}(2)$-valued. Therefore, we can use Γ^P as a connection not only on $\mathcal{Q}$, but also on $\mathcal{Q}'$ and try to extend it to an $\mathfrak{su}(3)$-valued connection on $\mathcal{P}'$. Once again, we illustrate the geometric situation, see figure 6.3.
Note that we have to perform the computations in the frames β_z. To this end, we will use an ansatz of the form

$$\Gamma = \Gamma^P + X_\mu \otimes \beta_z^\mu. \qquad (6.6.8)$$

Expressed in the coframes β_z the torsion of Γ^P reads:

$$T^5_{\ ab} = 2\varrho\,\eta^1_{\ ab} \quad \text{and} \quad T^b_{\ 5a} = -\frac{3}{2\varrho}\eta^{1b}_{\ \ a}. \tag{6.6.9}$$

From this we obtain the matrix equations on X as follows:

$$\begin{aligned} \big[I_i, X_\mu\big] &= f_{i\mu}^{\ \ \nu}\, X_\nu, \\ \big[X_5, X_a\big] &= \frac{3}{2\varrho}\,\eta^{1b}_{\ \ a}\, X_b + \eta^{3\ b}_{\ a}\,\frac{\mathrm{d}}{\mathrm{d}r} X_b, \\ \big[X_a, X_b\big] &= -\,2\,\varrho\,\eta^1_{\ ab}\, X_5 + \frac{2\varrho^2}{3}\,\eta^3_{\ ab}\,\frac{\mathrm{d}}{\mathrm{d}r} X_5 + \mathcal{N}_{ab}. \end{aligned} \tag{6.6.10}$$

As an explicit ansatz for the X_μ we consider

$$\begin{aligned} X_1(r) &= \psi(r)\,\big(\cos(\vartheta)\, I_1 - \sin(\vartheta)\, I_3\big), \\ X_3(r) &= \psi(r)\,\big(\cos(\vartheta)\, I_3 + \sin(\vartheta)\, I_1\big), \\ X_2(r) &= \psi(r)\,\big(\cos(\vartheta)\, I_2 + \sin(\vartheta)\, I_4\big), \\ X_4(r) &= \psi(r)\,\big(\cos(\vartheta)\, I_4 - \sin(\vartheta)\, I_2\big), \\ X_5(r) &= \chi(r)\, I_5, \end{aligned} \tag{6.6.11}$$

introducing a constant parameter $\vartheta \in [0, 2\pi]$. One can check that the frame-independence condition is satisfied for all $\vartheta \in [0, 2\pi]$ and that the 2-form part of $\mathcal{N}$ takes values in $\widetilde{W}(\mathcal{P}')$ exclusively. Explicitly, we have

$$\mathcal{N}_{\mu\nu} = \psi(r)^2\, f_{\mu\nu}^{\ \ \ i}\, I_i. \tag{6.6.12}$$

Thus, this is a valid ansatz for our reduction procedure for the instanton equations. Inserting this ansatz into the matrix equations leads to

$$\dot\chi = \frac{3}{\varrho^2}\,(\cos(\vartheta)^2 - \sin(\vartheta)^2)\,\psi^2, \tag{6.6.13}$$

$$\chi = \frac{2}{\varrho}\,\cos(\vartheta)\,\sin(\vartheta)\,\psi^2\,, \tag{6.6.14}$$

$$\cos(\vartheta)\,\dot\psi = -\frac{3}{2}\psi\,(\sin(\vartheta) - \varrho\,\cos(\vartheta)\,\chi)\,, \tag{6.6.15}$$

$$\sin(\vartheta)\,\dot\psi = \frac{3}{2}\psi\,(\cos(\vartheta) - \varrho\,\sin(\vartheta)\,\chi)\,. \tag{6.6.16}$$

By the dot we abbreviate the derivative with respect to r.
As it turns out, the equations lead to contradictions except for $\vartheta = \frac{\pi}{4}, \frac{3\pi}{4}$. For these values, we have $\frac{1}{\sqrt{2}} = \cos(\vartheta) = \pm\sin(\vartheta)$, and the first two equations yields

$\dot{\psi} = \dot{\chi} = 0$, while the last two coincide.
Apart from the trivial solution, this reduced system admits the solutions

$$\vartheta = \frac{\pi}{4} \quad : \qquad \psi = \pm 1, \quad \chi = \frac{1}{\varrho}, \tag{6.6.17}$$

$$\vartheta = \frac{3\pi}{4} \quad : \qquad \psi = \pm 1, \quad \chi = -\frac{1}{\varrho}. \tag{6.6.18}$$

Γ^P provided us with an instanton for the half-flat structure $\mathcal{P}'$ and even for the underlying $SU(2)$-structure $\mathcal{Q}'$. As the 2-form parts of their field strengths has vanishing 6-components, the above four instanton solutions are in fact even instantons for the $SU(2)$-structure $\mathcal{Q}'$. Therefore, by proposition 4.3.5, they also are instantons for the unrotated $SU(2)$-structure $\mathcal{Q}$ on $M^5 \times I$. Since the rotation (5.4.1) is constant as well, the instanton solutions obtained here do neither depend on r, nor have non-vanishing 6-components. Thus, these connections are pullbacks of connections living on the principal $SU(3)$-bundle $\iota_r{}^*\mathcal{P}'$ over M^5, where ι_r is the embedding of M^5 into $M^5 \times I$ as the slice at cone parameter $r \in I$. However, this means that we found new instantons on the Sasaki-Einstein 5-manifold. Note that as X is an Ad-equivariant 1-form on $\mathcal{P}'$, it can in general not be transported back to $\mathcal{P}$ via R_h, since the adjoint action of h preserves $SU(2)$ only, rather than $SU(3)$. Therefore, although Γ^P restricts to a connection on both $\mathcal{Q}$ and $\mathcal{Q}'$, the extensions of Γ^P to connections on $\mathcal{P}'$ found here do not restrict to connections on $\mathcal{P}$ and vice versa.

Γ^P is an instanton for both notions of the instanton condition mentioned before, and so are the new solutions obtained here, because they can be seen as lifts of instantons from M^5. This may be of interest since the original version (4.1.17) of the instanton condition is equivalent to

$$\gamma(F_A)(\epsilon) = 0, \tag{6.6.19}$$

where ϵ is the generalized Killing spinor defining the half-flat $SU(3)$-structure $\mathcal{P}'$ on the cylinder. Recall that in section 5.4 we argued that I may be any open interval in $\mathbb{R}$, and that the geometric structure may be extended to the boundary of $M^5 \times I$ smoothly. This also holds true for the solutions $\Gamma^P + X$ which we constructed in this section since they can be seen as living on every slice separately. Thus, we have obtained instantons on smooth, compact, 6-dimensional manifolds with half-flat $SU(3)$-structures. These, again, may well prove useful in string model building.

Chapter 7

Conclusions and Outlook

This thesis was primarily concerned with G-structures and instantons. In proposition 4.3.1, we found a certain class of deformation that take G-structures to different G-structures over the same manifold and with the same G. These were induced by the right-action with maps $h : M \to N_{GL(D,\mathbb{R})}(G)$ taking values in the normalizer of G in $GL(D, \mathbb{R})$.

We provided a bijection from connections compatible with the original G-structure to those preserving the deformed G-structure. Moreover, we classified the subset of these deformations for which the instanton bundles of the original and the deformed G-structures coincide. Although this turned out to be very helpful in the constructions of chapter 6, the investigation of these deformations of G-structures is not completed.

For example, one could ask whether there are more general transformations leaving the instanton bundle invariant, and whether the instanton bundle determines the G-structure up to these transformations. The answer to the first question might be to replace h by maps to $N_{GL(D,\mathbb{R})}(G)/(G\times G)$, as pointwise left- and right-multiplication of $h(x)$ by elements of G does not change the image of $R_h\,\mathcal{Q}$ and has no effect on the instanton bundle.

However, the question for the generic structure of the instanton moduli space of a G-structure appears to be much more intricate. Nevertheless, for classes of G-structures with the same instanton bundle the instanton moduli spaces coincide. The above considerations might provide a partial answer as to what these classes are.

In chapter 5, we introduced a formalism to construct 6-dimensional $SU(3)$-structure manifolds of topology $M^5\times I$ from 5-dimensional manifolds M^5, where M^5 is endowed with a one-parameter family of $SU(2)$-structures. We showed how the deformations of G-structures introduced in section 4.3 may be used to construct such families of $SU(2)$-structures from a given $SU(2)$-structure, which we took to be Sasaki-Einstein. We arrived at the same 6-dimensional spaces upon applying the rotations as a family and lifting the result, as well as upon lifting the constant $SU(2)$-structure followed by applying the rotation as a single transformation on $M^5\times I$.

By this procedure, we constructed a Kähler-torsion $SU(3)$-structure on the sine-cone over M^5, which is conformally equivalent to the Calabi-Yau metric cone over M^5. Employing a certain rotation on this sine-cone, we transformed this $SU(3)$-structure into the nearly Kähler structure on the sine-cone which had already been constructed in [24] by means of flow equations. Both of these geometries on the sine-cone approach the Calabi-Yau metric cone over M^5 in an appropriate large-volume limit. Furthermore, we constructed a two-parameter family of half-flat structures on cylinders utilizing another deformation of the constant lift of the Sasaki-Einstein $SU(2)$-structure to $M^5 \times I$. All of these spaces can be extended to compact conifolds, which are valuable candidates for internal spaces in string model building. Therefore, this procedure might prove useful in producing candidates for internal geometries in flux compactifications. They seem particularly useful for the construction of explicit solutions to several geometric equations, for they have a topologically trivial direction without having too simple a geometry.

The constructions in chapter 6 illustrated how the geometry of these spaces may be utilized in order to find solutions to geometric equations. Here we started with a formal treatment of the reduction procedure introduced in [1] for the instanton equations.
The results of section 4.3 enabled us to directly see that the lift of the canonical connection on the Sasaki-Einstein manifolds to $M^5 \times I$ provides us with an instanton for all the $SU(3)$-structures obtained before. From this we reproduced the instantons that one would have obtained by pulling back the instantons found on the metric cone in [2] along the conformal equivalence.
On the nearly Kähler space, employing the same ansatz as before we only found constant perturbations of the lift of the Sasaki-Einstein canonical connection. Via the results in section 4.3 we could infer that these solutions would be present for all the geometries we constructed in chapter 5. We constructed the canonical connection for the nearly Kähler structure and proved it to be an instanton using methods of [2]. This connection was in two different ways seen to split into an instanton for the $SU(2)$-structure that underlies the nearly Kähler structure and a part that we could use as an ansatz for a perturbation of this instanton. Here we could use the reduction procedure to give an independent proof that the nearly Kähler canonical connection is an instanton and to find another solution. Although the matrices which we used in the ansatz depended on the cone coordinate, the functions we used to parametrize our ansatz turned out to be constant for the solutions. Thus, these instantons appeared to be isolated in the sense that there are no additional solutions interpolating between them.
However, we observed that there is a direct correspondence between instanton extensions in sections 6.4.1 and 6.4.3, and transferring the respective solutions we obtained additional instanton extensions in both cases. These depend on the cone coordinate non-trivially.
On the half-flat cylinders over M^5 the instantons we found even turned out to stem from M^5 rather than the 6-dimensional space.
The methods of chapter 6 may well be generalized. One possibility would be to

consider more general gauge bundles $\mathfrak{B}$ that reduce to $\mathcal{Q}'$ in the terminology of section 6.1, as for example viable quiver bundles. It also seems plausible that there are general statements similar to proposition 5.1.1 that construct G_2-structures on $M^6 \times I$ from families of $SU(3)$-structures on M^6, and $Spin(7)$-structures on $M^7 \times I$ from families of G_2-structures on M^7. Combining such statements with the results of section 4.3 and section 6.1 might turn out to be as useful for constructing interesting explicit geometries and instanton solutions on them as it turned out to be for lifts of $SU(2)$-structures to $SU(3)$-structures.

It exceeded the scope of this work to employ the instantons we obtained in attempts to solve the field equations of heterotic supergravity, i. e. to find vacuum solutions of heterotic string theory by manners similar to those in [2–4]. Nevertheless, this could prove very fruitful, as the 6-manifolds we constructed are endowed with a (generalized) Killing spinor, a three form, which could be used as flux, a set of instantons and even a distinguished function r that might be useful in constructing the dilaton. In particular, on the nearly Kähler sine-cone, the canonical connection could be a valuable candidate for ∇^+. The gauge field could then be taken to coincide with this connection in the hope for simpler solutions, but one might as well try to use the other instantons that we found here. This seems to be a promising application of the findings of this thesis.

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Appendix A

Principal Fiber Bundles and Connections

A.1 Principal Fiber Bundles and Associated Bundles

Principal fiber bundles and connections are the central foundations of the mathematics used in this thesis. For this reason, we review the mathematical formalism of principal fiber bundles, associated bundles and connections. Throughout this thesis, we use the notation of [18], where all this formalism is presented beautifully. Thus, this section is a very condensed summary of the first three chapters of [18]. Proofs and details left out are to be found there or in other mathematics textbooks introducing the language of principal fiber bundles.

There are two crucial ingredients to this formalism, namely the notion of fiber bundles over a manifold and that of an action of a Lie group on a manifold. We start with the definition of a fiber bundle.

Definition A.1.1: *A **locally trivial fiber bundle** (or just fiber bundle) is a quadruple (E, π, M, F) with the following properties:*

(1) E, F and M are smooth manifolds.

(2) $\pi : E \to M$, $e \mapsto \pi(e)$ is a surjective submersion.

(3) There exists a family $\{(U_i, \phi_i)\}_{i\in\Lambda}$, where the U_i provide an open covering of M, and the ϕ_i are diffeomorphisms

$$\phi_i : \pi^{-1}(U_i) \to U_i \times F,\ e \mapsto \phi_i(e) \tag{A.1.1}$$

that preserve the base point. That is,

$$pr_{U_i} \circ \phi_i = \pi_{|\,\pi^{-1}(U_i)}. \tag{A.1.2}$$

E is then called the *total space* of the fiber bundle, whereas F is called the *typical fiber* of E. M is called the *base manifold* and the collection $\{(U_i, \phi_i)\}_{i\in\Lambda}$ is a *bundle atlas* of E. Any map satisfying the third axiom of the definition is called a *local trivialization* of E, since it diffeomorphically maps a subset of E to the trivial fiber bundle $U \times F$.

Apart from the trivial example of direct products, vector bundles are a very important class of fiber bundles. Here, the fiber F is a vector space, and the transition

maps between different local trivializations are linear automorphisms of this vector space.
In the following example we introduce a fiber bundle which is important throughout this text. It lies at the very fundamentals of the notion of a G-structure and yields the motivation for the abstract construction of associated vector bundles to be introduced later.

Example A.1.2: Consider a $\mathbb{K}$-vector bundle E of rank $k \in \mathbb{N}$ over a manifold M. $\mathbb{K}$ may be $\mathbb{R}$ or $\mathbb{C}$. Let us consider the set of bases of the fibers of E. They constitute the *frame bundle of* E. It is defined as the disjoint union

$$F(E) = \bigsqcup_{x \in M} \big\{ e_x = (e_{1|\,x}, \dots, e_{k|\,x}) \ \big|\ e_x \text{ is a basis of } E_x \big\}. \tag{A.1.3}$$

First, the prescription $\pi : F(E) \to M$, $\pi(e_x) \coloneqq x$ yields a surjective map. If we fix a basis e_x of a fiber E_x, every other basis of E_x is related to e_x by a unique $GL(k, \mathbb{K})$ transformation. Thus, if e_i is a local frame of E on some open subset $U_i \subset M$, we may consider the map

$$\phi_i : F(E)_{|\,U_i} \to U_i \times GL(k, \mathbb{K}),\ \ e'_{i|\,x} \mapsto (x, B), \tag{A.1.4}$$

where $B \in GL(k, \mathbb{K})$ is the unique matrix that induces the change of basis from $e_{i|\,x}$ to $e'_{i|\,x}$. That is,

$$e'_{i|\,x} = B^j{}_i\, e_{j|\,x}. \tag{A.1.5}$$

We can cover M by local frames of E. The induced transition maps between the ϕ_i are smooth with respect to the standard differentiable structures on $U_i \times GL(k, \mathbb{K})$. Thus, these maps induce a differentiable structure on $F(E)$ and provide a bundle atlas for $F(E)$. Therefore, this indeed is a locally trivial fiber bundle with typical fiber $GL(k, \mathbb{K})$.

On this bundle we can perform a global change of bases by acting on all possible bases with the same $B \in GL(k, \mathbb{K})$. This defines a map

$$\begin{aligned} R : & F(E) \times GL(k, \mathbb{K}) \to F(E), \\ & (e_x, B) \mapsto R_B\, e_x = \big(B^{j_1}{}_{i_1}\, e_{j_1|\,x}, \dots, B^{j_k}{}_{i_k}\, e_{j_k|\,x} \big) \\ & \eqqcolon \big((R_B\, e)_{1|x}, \dots, (R_B\, e)_{k|x} \big). \end{aligned} \tag{A.1.6}$$

Note that with this definition, $R_B \circ R_A = R_{AB}$. This is a special case of a right-action of a Lie group on a manifold, which we now define in general terms.

Definition A.1.3: *Let G be a Lie group and $\mathcal{P}$ a manifold. A* ***right-action*** *of G on $\mathcal{P}$ is a smooth map*

$$R : \mathcal{P} \times G \to \mathcal{P},\ \ (p, g) \mapsto R_g\, p \tag{A.1.7}$$

that satisfies

(1) *For every $g \in G$, the map $R_g : \mathcal{P} \to \mathcal{P}$, $p \mapsto R_g\, p$ is a diffeomorphism on $\mathcal{P}$.*
(2) $R_a \circ R_g = R_{ga} \quad \forall\, a, g \in G$.

A ***left-action*** *of G on $\mathcal{P}$ is a smooth map*

$$L : G \times \mathcal{P} \to \mathcal{P}, \ (g, p) \mapsto L_g\, p \tag{A.1.8}$$

that satisfies

(1) *For every $g \in G$ the map $L_g : \mathcal{P} \to \mathcal{P}$, $p \mapsto L_g\, p$ is a diffeomorphism on $\mathcal{P}$.*
(2) $L_a \circ L_g = L_{ag} \quad \forall\, a, g \in G$.

Heuristically speaking, a right-action is a Lie group anti-homomorphism from G to the group of diffeomorphisms on $\mathcal{P}$. A left-action is a group homomorphism of these groups.
Note that by introducing an inverse in the above example of a $GL(k, \mathbb{K})$-action on $F(E)$ one obtains a left-action instead of a right-action. One calls a group action *simply transitive* if for any $p, q \in \mathcal{P}$ there exists a unique $g \in G$ such that $R_g\, p = q$.

Also, the right-action of $GL(k, \mathbb{K})$ on $F(E)$ maps points of one fiber to points in the same fiber, i. e. it preserves the base point. Even more, given two particular points in the same fiber, there is precisely one element of $GL(k, \mathbb{K})$ whose action maps the one point to the other (namely the respective change of basis). Therefore, this right action is simply transitive on the fibers of $F(E)$. This is the motivating example of the following more abstract definition:

Definition A.1.4: *A* ***principal fiber bundle*** *is a tuple $(\mathcal{P}, \pi, M, G)$ having the following properties:*

(1) *$(\mathcal{P}, \pi, M, G)$ is a locally trivial fiber bundle, where the typical fiber G is a Lie group.*
(2) *There is a right-action $R : \mathcal{P} \times G \to \mathcal{P}$, $(p, g) \mapsto R_g\, p$ of G on $\mathcal{P}$ that is simply transitive on the fibers of $\mathcal{P}$, and that preserves base points, i. e.*

$$\pi \circ R_g = \pi \quad \forall\, g \in G. \tag{A.1.9}$$

(3) *There exists a G-equivariant bundle atlas $\{(U_i, \phi_i)\}_{i \in \Lambda}$ for $(\mathcal{P}, \pi, M, G)$. That is, for all $p \in \mathcal{P}$, $g \in G$, $i \in \Lambda$ we may write*

$$\phi_i(p) = \big(\pi(p),\, \psi_i(p)\big), \tag{A.1.10}$$

and this satisfies

$$\phi_i \circ R_g(p) = \big(\pi(p),\, r_g(\psi_i(p))\big), \tag{A.1.11}$$

where $r : G \times G \to G$, $(a, g) \mapsto r_g(a) = ag$ is the intrinsic right-multiplication on the Lie group.

The frame bundle of a vector bundle provides an example of how every vector bundle gives rise to a principal fiber bundle over the base space. However, this also works in the opposite direction. We illustrate this using again the frame bundle $F(E)$ of a vector bundle E over M.

Example A.1.5: In the above example, we constructed $F(E)$ from E by considering bases of E at every point of M. Every such basis induces an isomorphism of vector spaces

$$[e] : \mathbb{K}^k \to E_{\pi(e)},\ (v^1, \dots, v^k) \mapsto e_i\, v^i. \tag{A.1.12}$$

This is the natural way to construct a vector in E from a basis and linear coefficients, namely by linear combination. Therefore, one could define an element of E_x to be a tuple $X = (e, v)$, where $e \in F(E)_x$ and $v \in \mathbb{K}^k$.
This, however, is not a good definition, since the same element of E_x can be represented by several such tuples. The ambiguity is precisely the freedom in the choice of a basis of the fiber. In particular, we have

$$\begin{aligned} X &= (e, v) \\ &= e_i\, v^i \\ &= e_k\, B^k{}_i\, (B^{-1})^i{}_j\, v^j \\ &= \big(R_B\, e,\ \rho(B^{-1})(v)\big) \quad \forall\, B \in GL(k, \mathbb{K}), \end{aligned} \tag{A.1.13}$$

where ρ is the standard representation of $GL(k, \mathbb{K})$ on $\mathbb{K}^k$. All these tuples are a decomposition of *the same* vector with respect to different bases of the fiber. Hence, the correct notion of a vector in E_x is the entity of all these tuples. That is, vectors in E_x are equivalence classes

$$[e, v] = [R_B\, e, \rho(B^{-1})(v)] \quad \forall\, B \in GL(k, \mathbb{K}). \tag{A.1.14}$$

These are just all possible ways to decompose a given vector in arbitrary bases of the fiber.

This concept of gluing vectors from bases and coefficients can be generalized to some extent. We actually neither need the principal bundle to be a frame bundle, nor do we need the structure group to be a matrix group, nor the fiber to be a vector space. Nevertheless, the previous example illustrates and motivates the following definition:

Definition A.1.6: *Let $(\mathcal{P}, \pi, M, G)$ be a principal bundle over M, F a smooth manifold and $\rho : G \times F \to F$, $(g, v) \mapsto \rho(g)(v)$ a left-action of G on F. From this* *data one can construct the* ***associated bundle***

$$E = \mathcal{P} \times_{(G,\rho)} F := (\mathcal{P} \times F)/\sim, \tag{A.1.15}$$

where

$$(p, v) \sim (p', v') \quad \Leftrightarrow \quad \exists\, g \in G : (p', v') = \big(R_g\, p,\ \rho(g^{-1})(v)\big). \tag{A.1.16}$$

As explained above, in the case of F being a k-dimensional $\mathbb{K}$-vector space the equivalence relation can be understood as to divide out precisely the ambiguity in representing the same vector with respect do different bases. E can then be shown to be a smooth $\mathbb{K}$-vector bundle of rank k over M.
The link to the formalism of components, which is used in most physics communications, is given by the *fiber isomorphism.* Be $p \in \mathcal{P}_x$ and E as defined above. The fiber isomorphism is the map

$$[p] : F \to E_x,\ v \mapsto [p, v]. \tag{A.1.17}$$

The components, or the local representation, of an element X of E_x is then just given by

$$v = [p]^{-1}(X)\ \in F. \tag{A.1.18}$$

If $\gamma \in \Gamma(E)$ is a section of an associated bundle and $s \in \Gamma_U(\mathcal{P})$ is a local section of $\mathcal{P}$ over some $U \subset M$, the local representation of γ with respect to s is

$$v : U \to F,\ v(x) = [s(x)]^{-1}(\gamma_x), \tag{A.1.19}$$

which is a map from U to the space F. In the other direction, every section $\gamma \in \Gamma(E)$ is locally of the form

$$\gamma_x = [s(x), v(x)] \tag{A.1.20}$$

for some local section s of $\mathcal{P}$ and a locally defined map $v : U \to F$. For $F = V$ a vector space, it is these locally defined maps with values in vector spaces that are used commonly in the physics literature.

There is yet another view on sections of associated vector bundles. Let us again assume we are given a section $\gamma \in \Gamma(E)$, where E is a vector bundle associated to $(\mathcal{P}, \pi, M, G)$ via the G-representation ρ on V. We may ask ourselves the question whether there is a map $\hat{\gamma} : \mathcal{P} \to V$, $p \mapsto \hat{\gamma}(p)$ such that the local representation of γ with respect to any local section $s \in \Gamma_U(\mathcal{P})$ coincides with the pullback of $\hat{\gamma}$ by s. This way, we could transform the complete geometry of sections of non-trivial associated vector bundles to maps from $\mathcal{P}$ to certain vector spaces, which are easier to handle.
As it turns out, this is possible not only for sections of E, but even for differential forms on M with values in E. In order to formulate this assertion precisely, we first need some further notions.

Definition A.1.7: *Let $(\mathcal{P}, \pi, M, G)$ be a principal bundle and $E = \mathcal{P} \times_{(G,\rho)} V$ an associated vector bundle. First, the collection of tangent spaces to the fibers of $\mathcal{P}$ is a smooth vector-subbundle of $T\mathcal{P}$. It is called the* ***vertical tangent bundle*** *$T^v\mathcal{P}$ of $\mathcal{P}$. As π projects every fiber to its base point, we have*

$$T^v\mathcal{P} = ker(\pi_*) \subset T\mathcal{P}. \tag{A.1.21}$$

Now, let $\omega \in \Omega^k(\mathcal{P}, V)$ be a k-form on $\mathcal{P}$ with values in the vector space V. We say that ω is a ***horizontal*** *k-form if it vanishes whenever one inserts a vertical vector. We denote the set of horizontal V-valued k-forms on $\mathcal{P}$ by $\Omega^k_{hor}(\mathcal{P}, V)$.*
Finally, we say that $\omega \in \Omega^k(\mathcal{P}, V)$ is ***of type*** *ρ if*

$$R_g{}^*\,\omega = \rho(g^{-1}) \circ \omega \quad \forall\, g \in G, \tag{A.1.22}$$

and we denote the set of V-valued k-forms of type ρ on $\mathcal{P}$ by $\Omega^k(\mathcal{P}, V)^{(G,\rho)}$.

The complete geometry of E-valued differential forms on M is indeed encoded in horizontal V-valued differential forms of type ρ on $\mathcal{P}$, as the following proposition states:

Proposition A.1.8: *In the above situation, there is an isomorphism*

$$\Psi : \Omega^k(M, E) \to \Omega^k_{hor}(\mathcal{P}, V)^{(G,\rho)},\ \mu \mapsto \Psi(\mu), \tag{A.1.23}$$

where for any p in the fiber over x, i. e. with $\pi(p) = x$,

$$\Psi(\mu)_p = [p]^{-1} \circ \mu_x \circ \pi_{*|p} = [p]^{-1} \circ (\pi^*\mu)_p\,. \tag{A.1.24}$$

This implies that for a local section $s \in \Gamma_U(\mathcal{P})$ we have

$$(s^*\Psi(\mu))_x = [s(x)]^{-1} \circ ((\pi \circ s)^*\mu)_{|s(x)} = [s(x)]^{-1} \circ \mu_x, \tag{A.1.25}$$

which is precisely the local representation of the bundle-valued form μ. Note that for any local section $s \in \Gamma_U(\mathcal{P})$,

$$\mu_{|x} = [s(x)] \circ (s^*\Psi(\mu))_x = \big[s(x), s^*\Psi(\mu)_x\big]. \tag{A.1.26}$$

This glues together the globally defined, bundle-valued form μ from the local pull-backs of its pendant on $\mathcal{P}$. We illustrate this in the following, final example of this section.

Example A.1.9: Let us consider a vector field $X \in \Gamma(TM) = \Omega^0(M, TM)$. Recall that $TM = F(TM) \times_{(GL(D,\mathbb{R}),\rho)} \mathbb{R}^D$ with the standard representation of $GL(D, \mathbb{R})$ on $\mathbb{R}^D$, and that the elements of $F(TM)$ are the bases of the fibers of TM.

The fiber isomorphism induced by a given basis is just the decomposition of a vector with respect to that basis. That is, if $e \in \Gamma_U(F(TM))$ a local frame of TM, the local representation of X can be read off from

$$X_x = X^\mu(x)\, e_{\mu|x} = \big[e(x), (X^1(x), \ldots, X^D(x))\big] \quad \forall\, x \in U. \tag{A.1.27}$$

Further, since $X \in \Omega^0(M, TM)$, by the above proposition there exists a pendant of X on $F(TM)$, which we denote by $\Psi(X) \in \Omega^0_{hor}(F(TM), \mathbb{R}^D)^{(GL(D,\mathbb{R}),\mathbb{R}^D)}$. It is constructed such that its pullback along $e : U \to F(TM)$ gives precisely the same local representation

$$(e^*\, \Psi(X))_x = \big(X^1(x), \ldots, X^D(x)\big) \quad \forall\, x \in U. \tag{A.1.28}$$

Thus, one may carry out local calculations completely with the local representations $(X^1, \ldots, X^D) : U \to \mathbb{R}^D$, but one has to keep in mind that these still have to form globally well-defined objects. That is, the local representations depend on the choice of local sections of principal bundles, and this has to be accounted for carefully. Usually, this is just the transformation behavior of tensor components, but in the main text we will encounter an example where this leads to a more complex interplay of several transformation laws, whose balancing imposes nontrivial conditions (see sect. 6.1).
In that case, we deal with a two-form with values in the *adjoint bundle* of a principal bundle. This is the associated bundle

$$Ad(\mathcal{P}) := \mathcal{P} \times_{(G,Ad)} \mathfrak{g}. \tag{A.1.29}$$

For example, consider $F \in \Omega^2(M, Ad(\mathcal{P}))$. Then we have $\Psi(F) \in \Omega^2_{hor}(\mathcal{P}, \mathfrak{g})^{(G,Ad)}$, and its local representation with respect to a section $s \in \Gamma_U(\mathcal{P})$ is

$$s^*\Psi(F) \in \Omega^2(U, \mathfrak{g}). \tag{A.1.30}$$

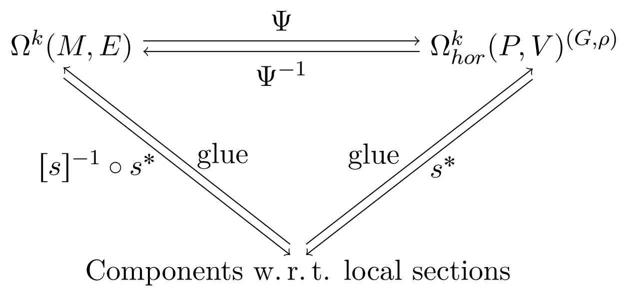
If we additionally decompose this locally defined two-form with respect to a local frame $e \in \Gamma_U(F(TM))$, its two-form part is decomposed into its components with respect to this basis. Thus, the local representative becomes a map

$$\big((s^*\Psi(F))_{ij}\big) : U \to \Lambda^2\mathbb{R}^{D*} \otimes \mathfrak{g}. \tag{A.1.31}$$

Note that, in general, this requires two local sections s and e of the principal bundles $\mathcal{P}$ and $F(TM)$, respectively.

In conclusion, the simplest relation between the local expressions and global forms is the fact that the local representations are pullbacks of horizontal, vector-space-valued forms on $\mathcal{P}$ with a certain transformation behavior. Both points of view are

able to overcome the non-triviality of associated vector bundles that, for example, F takes values in.
The relations between the three ways to view bundle-valued forms on M are depicted the following diagram:



(A.1.32)

A.2 Connections and Covariant Derivatives

To keep this chapter at least a little compact, we do not elaborate on the physical motivation of connections and their emergence in field theories, but come to their mathematical construction directly.
Consider a principal bundle $(\mathcal{P},\pi,M,G)$. As π is a submersion, the fibers are submanifolds of $\mathcal{P}$, i. e. $\mathcal{P}$ is foliated into leaves of type G by π. This gives rise to the vertical tangent bundle $T^v\mathcal{P}\subset T\mathcal{P}$.
In contrast to this, there is, in general, no foliation of $\mathcal{P}$ into leaves of type M transversal to the first one. (If there was such an embedding of M into $\mathcal{P}$, this would yield a global section of $\mathcal{P}$, whence, as one can show, $\mathcal{P}$ would be trivial).
Nevertheless, there may be a geometric distribution $T^h\mathcal{P}$ complementary to $T^v\mathcal{P}$ in the sense that

$$T\mathcal{P}=T^v\mathcal{P}\oplus T^h\mathcal{P}. \tag{A.2.1}$$

We call such a distribution *horizontal.* This can be chosen to be integrable if and only if $\mathcal{P}$ is trivial.
There are other ways to characterize such a splitting of $T\mathcal{P}$. One of them is to choose a subspace complementary to $T^v_p\mathcal{P}$ in $T_p\mathcal{P}$ by writing it as the kernel of a linear map. As the dimension of its kernel must then be D, its image must have dimension $dim(G)$. This is achieved by considering e. g. $A_p: T_p\mathcal{P}\to\mathfrak{g}$. Thereby, we obtain a $\mathfrak{g}$-valued 1-form on $\mathcal{P}$.

Definition A.2.1: *Be $(\mathcal{P},\pi,M,G)$ a principal bundle over M. Consider the right action on $\mathcal{P}$ generated by an element of $\mathfrak{g}$. As under the right-action every point* *moves smoothly, this will define the vector field tangent to these motions:*

$$\varphi : \mathfrak{g} \to \Gamma(T^v\mathcal{P}),\ \varphi(\xi)_p := \frac{\mathrm{d}}{\mathrm{d}t}_{|0}\ R_{\exp(t\,\xi)}\, p. \tag{A.2.2}$$

We call $\varphi(\xi)$ the ***fundamental vector field*** *generated by ξ.*

There are the following equivalent definitions of a ***connection*** *on $\mathcal{P}$.*

(1) *A connection on $\mathcal{P}$ is a right-invariant, horizontal geometric distribution $T^h\mathcal{P}$ on $\mathcal{P}$.*

(2) *There exists a one-to-one correspondence between connections on $\mathcal{P}$ and one-forms $A \in \Omega^1(\mathcal{P}, \mathfrak{g})^{(G,Ad)}$ that additionally satisfy*

$$A(\varphi(\xi)_p) = \xi \quad \forall\, \xi \in \mathfrak{g},\, p \in \mathcal{P}. \tag{A.2.3}$$

The correspondence is given by

$$T^h\mathcal{P} = ker(A). \tag{A.2.4}$$

A is called the ***connection form*** *or* ***gauge field*** *for the corresponding connection, but often we refer to it just as a* ***connection*** *on $\mathcal{P}$. We denote the set of connection forms on $\mathcal{P}$ by $\mathcal{C}(\mathcal{P})$.*

(3) *Let $\{(U_i, s_i)\}_{i\in\Lambda}$ be a covering of M by local sections of $\mathcal{P}$. Then, for every non-empty $U_i \cap U_j = U_{ij} \neq \varnothing$ there is a transition map $g_{ij} : U_{ij} \to G$, $x \mapsto g_{ij}(x)$ such that*

$$s_i(x) = R_{g_{ij}(x)}\, s_j(x). \tag{A.2.5}$$

There exists a one-to-one correspondence between $A \in \mathcal{C}(\mathcal{P})$ and collections $A_i \in \Omega(U_i, \mathfrak{g})$ subject to the compatibility relations

$$A_i = Ad({g_{ij}}^{-1}) \circ A_j + {g_{ij}}^* \mu_G, \tag{A.2.6}$$

where μ_G is the Maurer-Cartan form of the Lie group G. The link to (2) is given by $A_i = s_i^*\, A$.

In general, the exterior differential of a horizontal form on $\mathcal{P}$ will not be horizontal anymore. However, a connection provides the tool necessary to deform the differential such that it preserves horizontality. One simply projects $\mathrm{d}\omega$ to its part acting on $T^h\mathcal{P} = ker(A)$.

Definition A.2.2: *The* ***covariant exterior differential****, or just* ***covariant differential****, of $\omega \in \Omega^k(\mathcal{P}, V)$ with respect to the connection $A \in \mathcal{C}(\mathcal{P})$ is defined as*

$$D_A\, \omega = \mathrm{d}\omega \circ pr_{ker(A)} \in \Omega^{k+1}_{hor}(\mathcal{P}, V). \tag{A.2.7}$$

That is, applied to $X_0, \ldots, X_k$ we have

$$D_A\, \omega = \mathrm{d}\omega\big(pr_{ker(A)}(X_1), \ldots, pr_{ker(A)}(X_k)\big). \tag{A.2.8}$$

This derivative can also be written in terms of the connection form. It then turns out that this modified exterior differential also preserves the type of horizontal forms.

Proposition A.2.3: *The covariant differential of* $\omega \in \Omega^k_{hor}(\mathcal{P}, V)^{(G,\rho)}$ *with respect to the connection* $A \in \mathcal{C}(\mathcal{P})$ *is*

$$D_A\,\omega = \mathrm{d}\omega + \rho(A) \overset{\circ}{\wedge} \omega \in \Omega^{k+1}_{hor}(\mathcal{P}, V)^{(G,\rho)}, \tag{A.2.9}$$

where for bases $\{I_a\}$ *of* $\mathfrak{g}$ *and* $\{v_n\}$ *of* V *one defines*

$$\rho(A) \overset{\circ}{\wedge} \omega \coloneqq \rho(I_a)(v_n) \otimes A^a \wedge \omega^n. \tag{A.2.10}$$

The covariant differential of $\mu \in \Omega^k(M, E)$ *for* $E = \mathcal{P} \times_{(G,\rho)} V$ *is defined as*

$$(\mathrm{d}_A\,\mu)_x = \big[s(x), s^*(D_A\,\Psi(\mu))_x\big], \tag{A.2.11}$$

where $s \in \Gamma_U(\mathcal{P})$ *is a local section.*
For the local representation of $D_A\,\omega$ *we thus have*

$$s^*(D_A\,\omega) = \mathrm{d}\,(s^*\omega) + \rho(s^*A) \overset{\circ}{\wedge} s^*\omega. \tag{A.2.12}$$

These are local, V*-valued* 1*-forms. After splitting off the basis of the vector space we end up with*

$$(s^*(D_A\,\omega))^a = \mathrm{d}\,(s^*\omega)^a + \rho(s^*A)^a{}_b \wedge (s^*\omega)^b. \tag{A.2.13}$$

This is the form of the covariant differential of a bundle-valued k-form that is often used in the physics literature in the context of non-Abelian gauge theories. For the Levi-Civita connection on TM we have $\mathcal{P} = F(TM)$ and we can use a local frame $e \in \Gamma_U(F(TM))$ as a section of *both* the frame bundle and the gauge bundle *simultaneously*. This fact will be important in section 6.1.
In this case,

$$\rho(s^*A)^c{}_d \eqqcolon \omega_a{}^c{}_d\,\beta^a, \tag{A.2.14}$$

and the components $\omega_a{}^c{}_d$ are called the Christoffel symbols of the connection with respect to the frame e.

The probably most important quantity derived from a connection is its field strength.

Definition A.2.4: *The* ***field strength*** *of a connection* $A \in \mathcal{C}(\mathcal{P})$ *is*

$$F^A = D_A\,A = \mathrm{d}A + \frac{1}{2}\,ad(A) \overset{\circ}{\wedge} A \in \Omega^2_{hor}(\mathcal{P}, \mathfrak{g})^{(G,Ad)}\ . \tag{A.2.15}$$

Its bundle-valued pendant on M *is*

$$F_A = \Psi^{-1}(F^A)\ \in \Omega^2(M, Ad(\mathcal{P})). \tag{A.2.16}$$

Its local representations $s^*F^A \in \Omega^2(U, \mathfrak{g})$ *can be written in terms of the local representations* $s^*A \in \Omega^1(U, \mathfrak{g})$ *of* A *by*

$$(s^*F^A) = \mathrm{d}(s^*A) + \frac{1}{2}\, ad(s^*A) \stackrel{\circ}{\wedge} s^*A. \tag{A.2.17}$$

We list some important properties of this field strength two-form.

Proposition A.2.5: *Let* $(\mathcal{P}, \pi, M, G)$ *be a principal fiber bundle over* M*,* $A \in \mathcal{C}(\mathcal{P})$ *a connection on* $\mathcal{P}$ *and* F_A *its field strength. Then the following assertions hold true:*

(1) The field strength satisfies the Bianchi identity

$$D_A\, F^A = 0. \tag{A.2.18}$$

(2) The square of the covariant derivative on $\omega \in \Omega^k_{hor}(\mathcal{P}, V)^{(G,\rho)}$ *is*

$$D_A\, D_A\, \omega = \rho(F^A) \stackrel{\circ}{\wedge} \omega \in \Omega^{k+2}_{hor}(\mathcal{P}, V)^{(G,\rho)}. \tag{A.2.19}$$

(3) The field strength satisfies

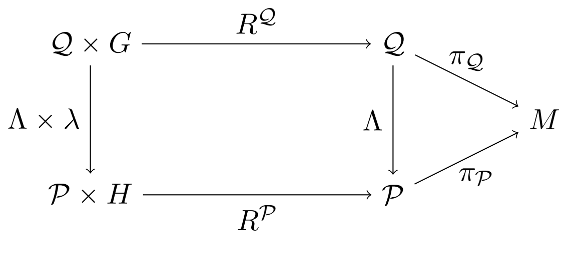
$$pr_{T^v\mathcal{P}}\big([X, Y]\big) = \varphi\big(F^A(X, Y)\big) \quad \forall\, X, Y \in \Gamma(T^h\mathcal{P}). \tag{A.2.20}$$

Therefore, $\mathcal{P}$ *admits a flat connection (i. e.* $F^A = 0$*) if and only if the geometric distribution* $T^h\mathcal{P}$ *is integrable.*

Finally, we introduce the notion of reductions of principal fiber bundles.

Definition A.2.6: *A reduction of a principal fiber bundle* $(\mathcal{P}, \pi_{\mathcal{P}}, M, H)$ *is a triple* $(\lambda, \Lambda, \mathcal{Q})$*, where* $(\mathcal{Q}, \pi_{\mathcal{Q}}, M, G)$ *is a principal* G*-bundle over* M*,* $\lambda : G \to H$*,* $g \mapsto \lambda(g)$ *is a homomorphism of Lie groups and* $\Lambda : \mathcal{Q} \to \mathcal{P}$*,* $q \mapsto \Lambda(q)$ *is a homomorphsim of principal fiber bundles, i. e. a map satisfying* $\Lambda \circ R^{\mathcal{Q}}_g = R^{\mathcal{P}}_{\lambda(g)} \circ \Lambda$*.*

This is often depicted as the following commutative diagram:

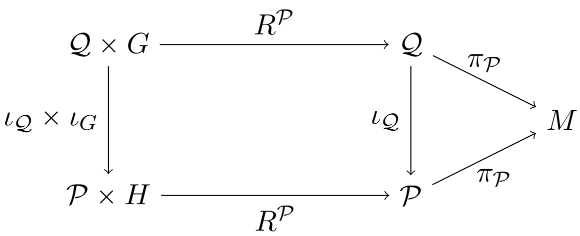


(A.2.21)

One can then show that connections on $\mathcal{Q}$ induce connections on $\mathcal{P}$. That is, to every $A \in \mathcal{C}(\mathcal{Q})$ there exists an $\tilde{A} \in \mathcal{C}(\mathcal{P})$ such that

$$\Lambda^*\tilde{A} = \lambda_* \circ A \quad \text{and} \quad \Lambda^* F^{\tilde{A}} = \lambda_* \circ F^A. \tag{A.2.22}$$

In general, the converse does not hold true, since $\Lambda^* \tilde{A}$ does not necessarily take values in $\lambda_*(\mathfrak{g}) \subset \mathfrak{h}$ for a generic $\tilde{A} \in \mathcal{C}(\mathcal{P})$. This is basically the foundation of the classification of G-structures in section 2.2 and also plays an important role throughout chapter 6. In this thesis we make extensive use of the special type of bundle reductions where $\mathcal{Q}$ is an embedded principal subbundle of $\mathcal{P}$ and λ is an embedding of a Lie subgroup $G \subset H$ into H, i. e.



(A.2.23)

Note that, in this situation, the right-action of G on $\mathcal{Q}$ is just the restriction of the right-action of H on $\mathcal{P}$ to the subgroup $G \subset H$ and the principal subbundle $\mathcal{Q} \subset \mathcal{P}$. The extension $\tilde{A}$ of a given $A \in \mathcal{C}(\mathcal{Q})$ is just the extension to the ambient principal bundle. Thus, it satisfies $\iota^* \tilde{A} = \tilde{A}_{|\mathcal{Q}} = A$.

As a final comment, note that such reductions can be constructed from a given principal bundle. That is, for any homomorphism of Lie groups $\sigma : G \to H$, $g \mapsto \sigma(g)$ one can construct the associated H-bundle

$$\mathcal{P} \coloneqq \mathcal{Q} \times_{(G, \ell \circ \sigma)} H, \tag{A.2.24}$$

where $\ell : H \times H \to H$ is the left-multiplication on H. It can be shown that $\mathcal{P}$ is a principal H-bundle, called the σ-extension of $\mathcal{Q}$, and that, in this situation, $\mathcal{Q}$ is a bundle reduction of $\mathcal{P}$.

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